Enumeration of rectangulations and corner polyhedra

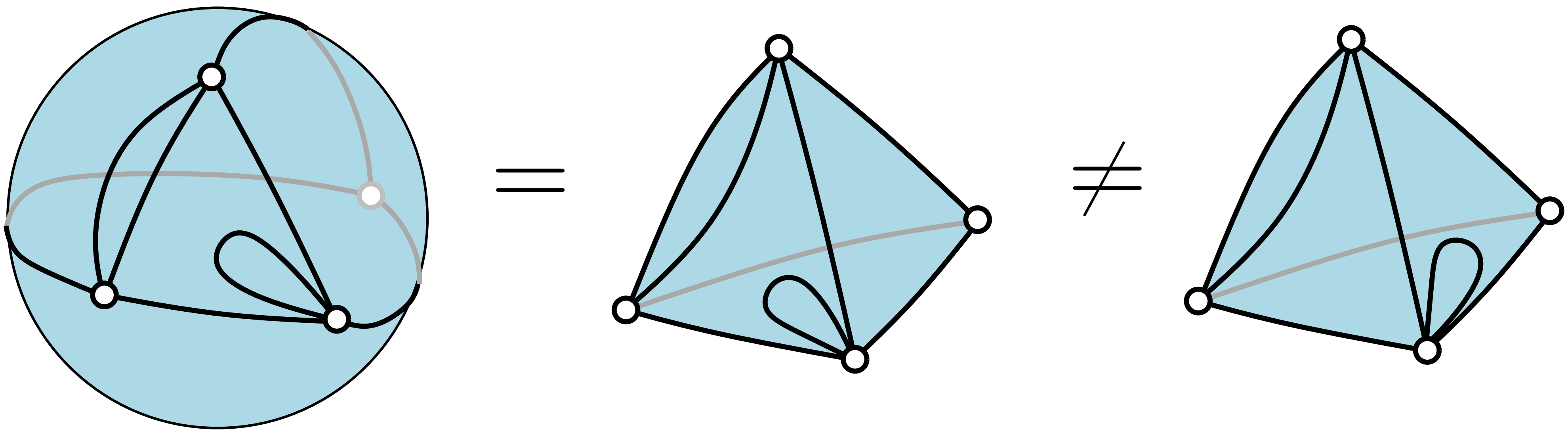
Éric Fusy (LIGM, Univ. Gustave Eiffel)

Joint work with Erkan Narmanli and Gilles Schaeffer

FPSAC'23, UC Davis

Planar maps

Def. Planar map = connected graph embedded on the sphere



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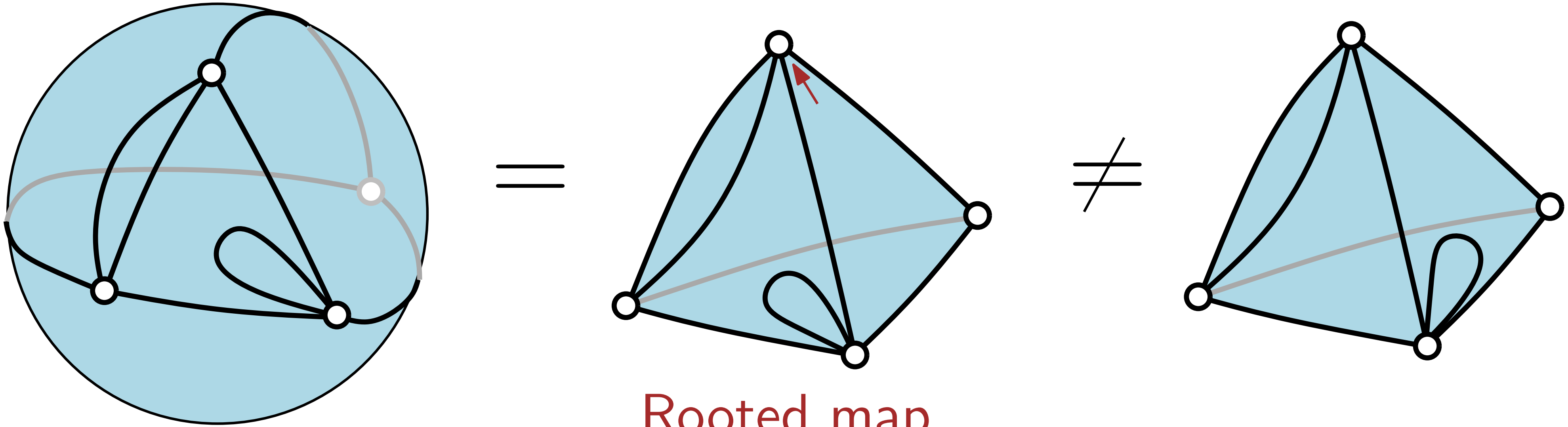

Rooted map
= map with marked corner

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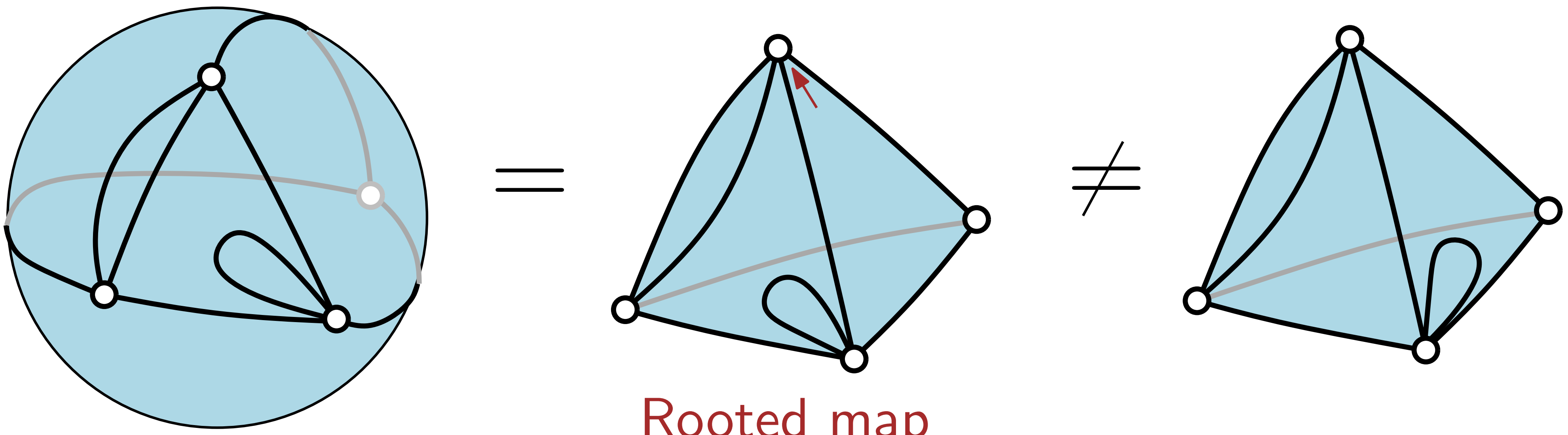
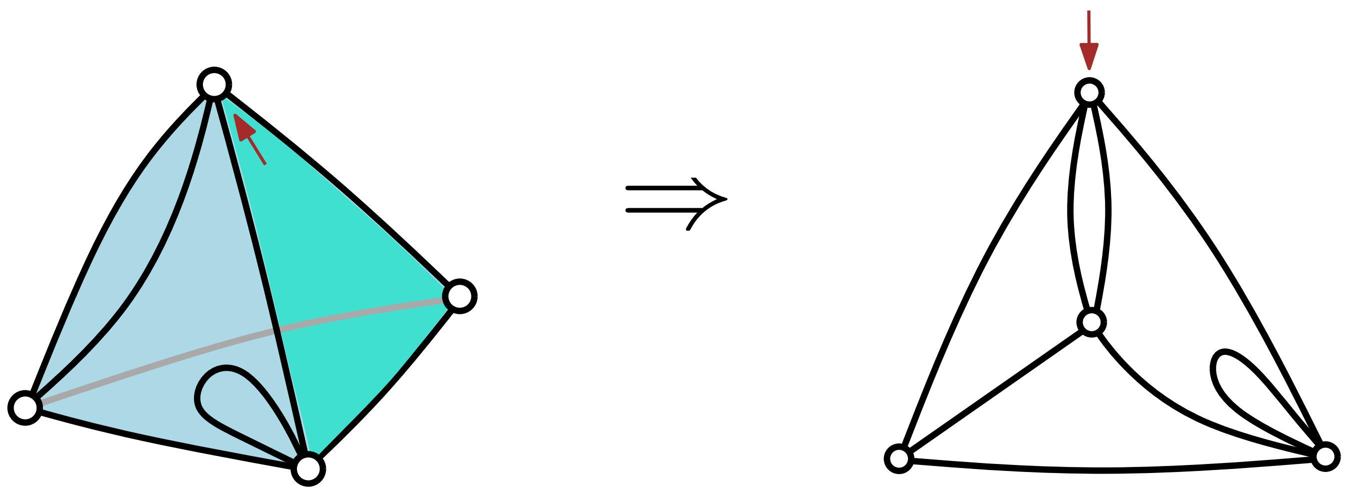


Rooted map
= map with marked corner

Easier to draw in the plane (choosing root-face to be the outer face)



Universality properties for planar maps

- Nice counting formulas for many natural families [Tutte'60s]

e.g. $$\#\text{ rooted maps } n \text{ edges} = \frac{2\cdot 3^n}{n!(n+2)!} = \frac{2}{n+2}3^n\mathrm{Cat}_n$$

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  [Bousquet-Mélou-Jehanne'06]

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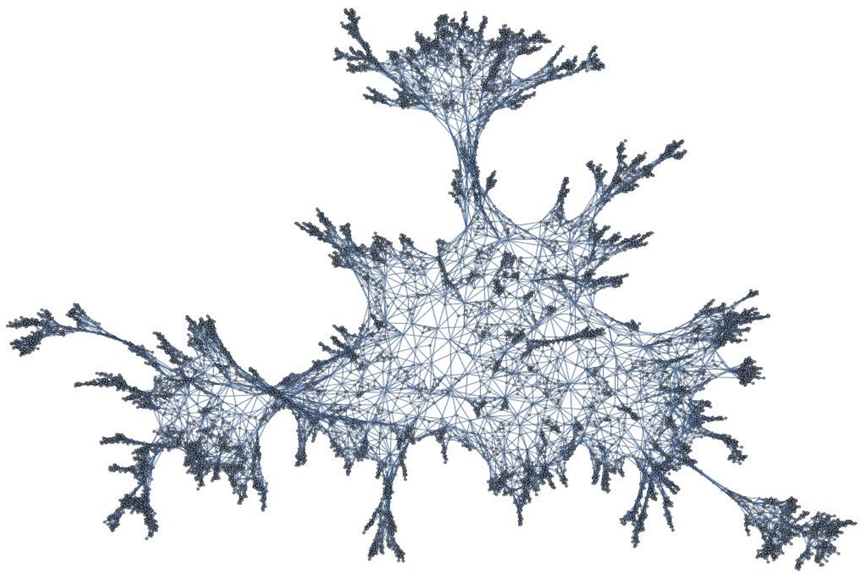
- Universal scaling limit (Brownian map) for random planar maps (rescaling distances by $n^{1/4}$)

  [Chassaing,Schaeffer'04]
  [Le Gall'13, Miermont'13]



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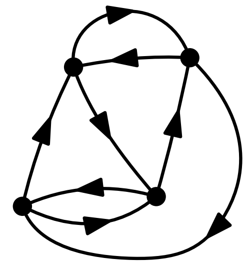
Decorated planar maps

Decorated planar map = planar map + structure

(Ising model, proper coloring, Potts model,
spanning tree, spanning forest, specific orientations,...)



4-regular map
+ Eulerian orientation

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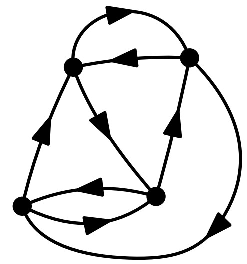

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• new bijections & counting behaviours compared to "pure" planar maps

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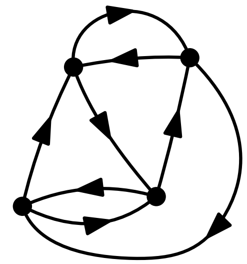

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link to "central charge"

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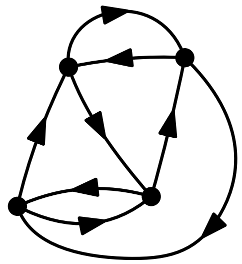

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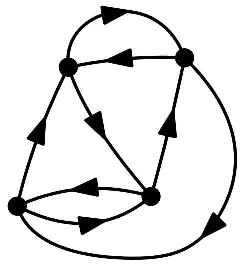

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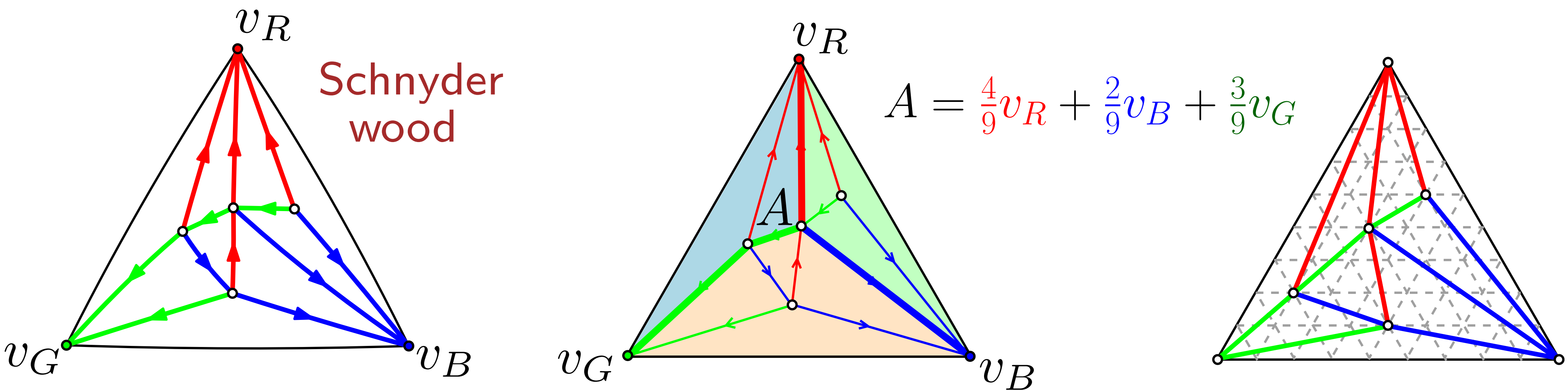
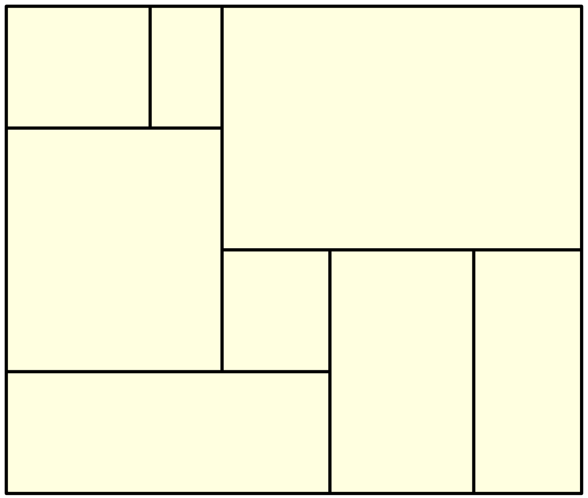
- Some of these structures give nice geometric representations of maps

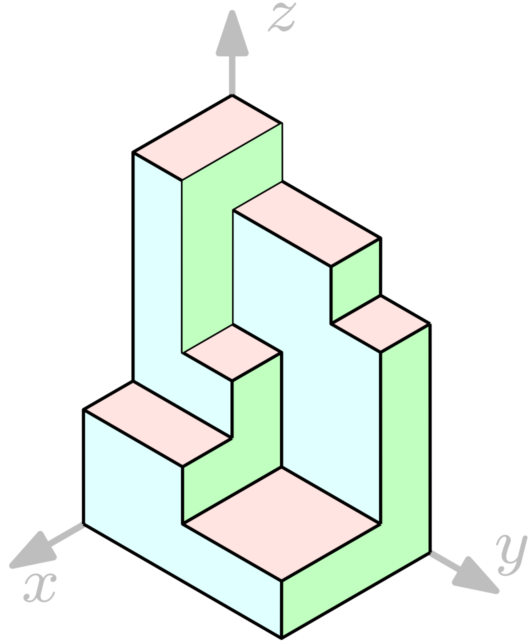


This talk

We consider two types of geometric representations



rectangulations



corner polyhedra

- Link to decorated planar maps & bijections to walks
- Exact enumeration
- Asymptotic enumeration

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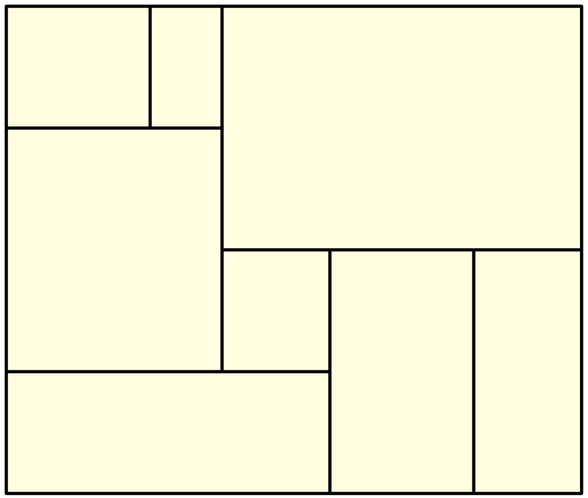

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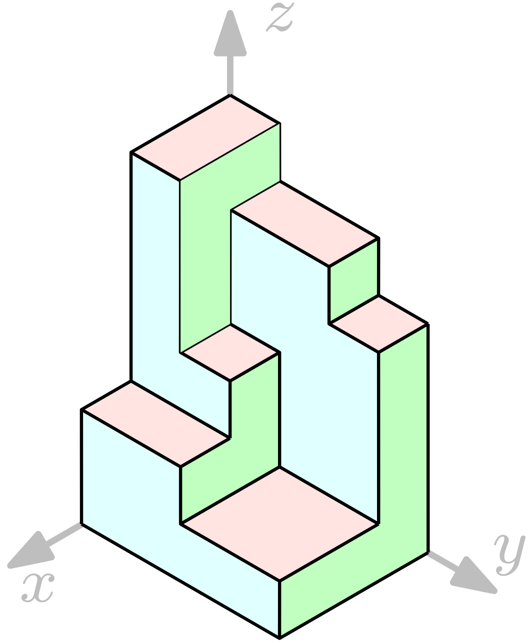


corner polyhedra
(missed talk at FPSAC'22)

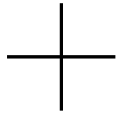
- Link to decorated planar maps & bijections to walks
- Exact enumeration
- Asymptotic enumeration

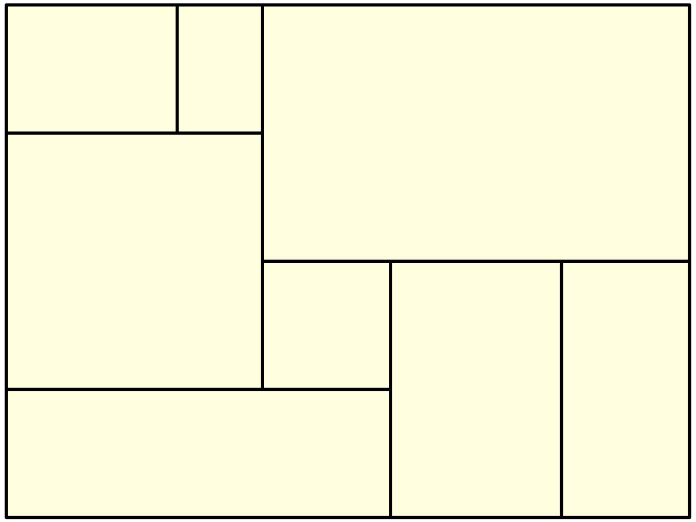
Rectangulations

(bicolored contact-systems)

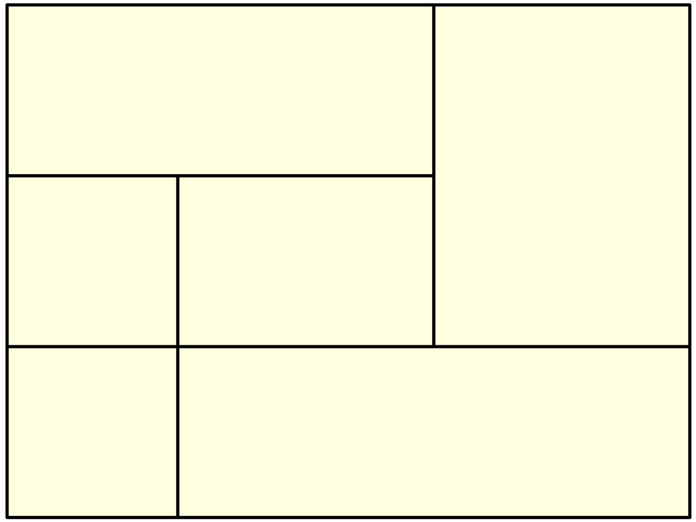
Rectangulations

Rectangulation = tiling of a rectangle by rectangles

Called "generic" if no 

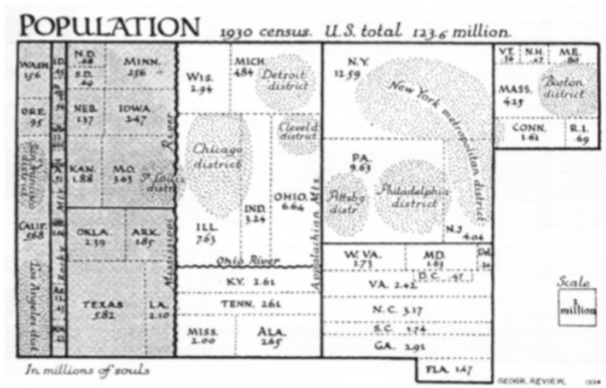


Generic



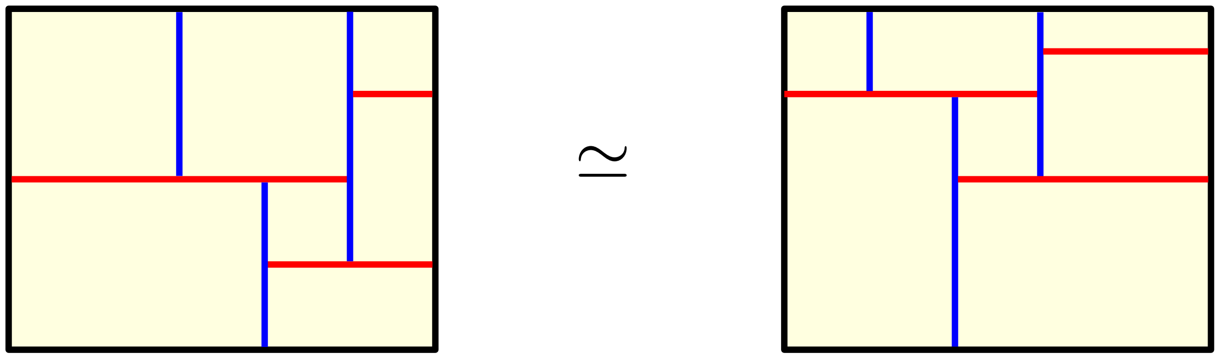
Not generic

used in "cartogram" representations



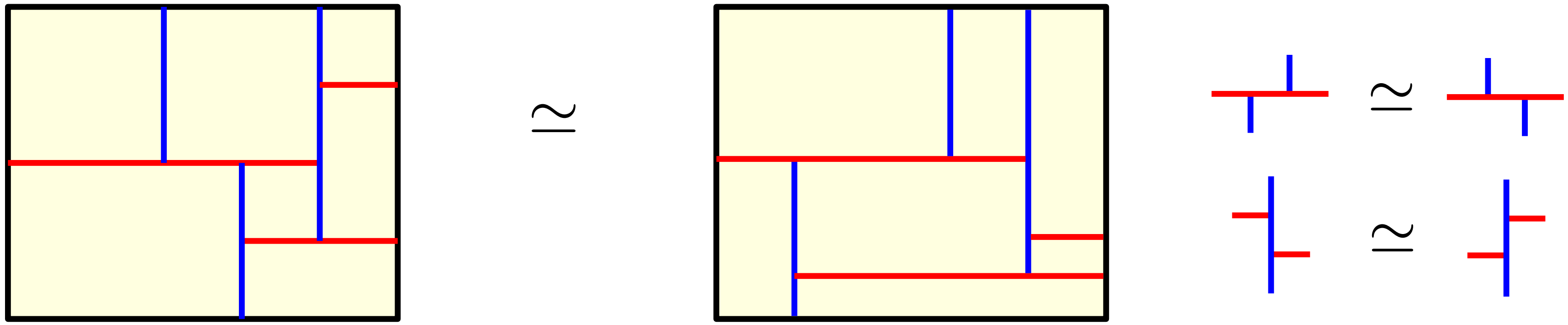
Two types of equivalences

Strong



(order of contacts along each maximal segment is preserved)

Weak



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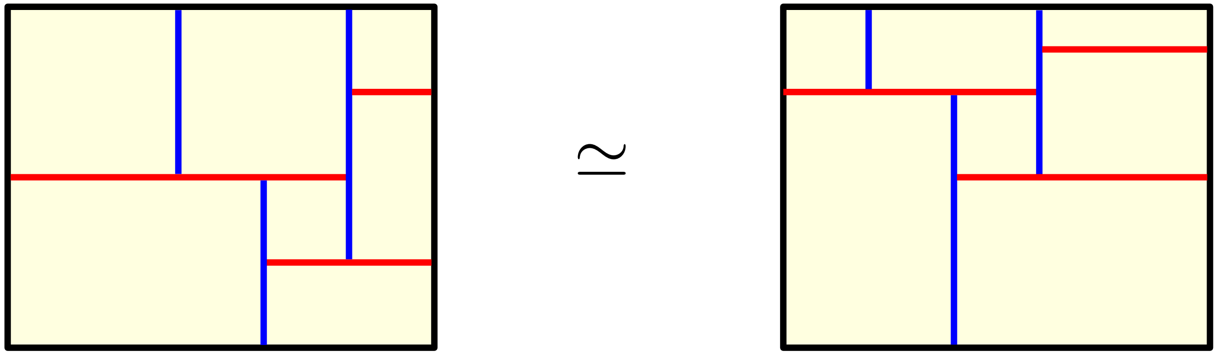

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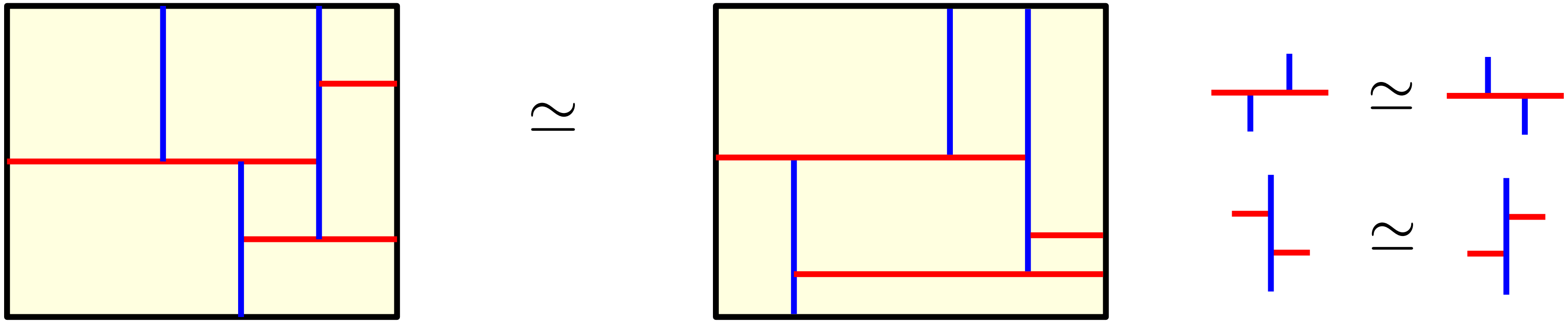

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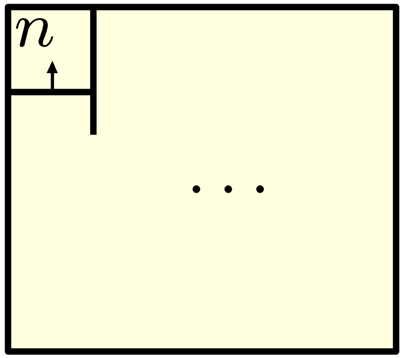
$w_n =$ # weak equivalence classes with n regions

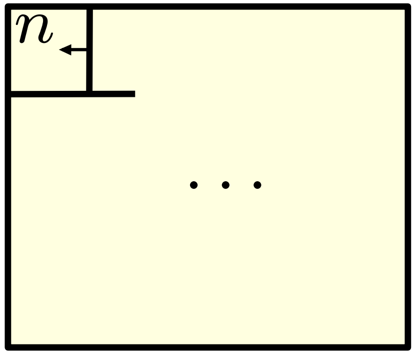
$s_n =$ # strong equivalence classes with n regions

Weak equivalence class: shelling order

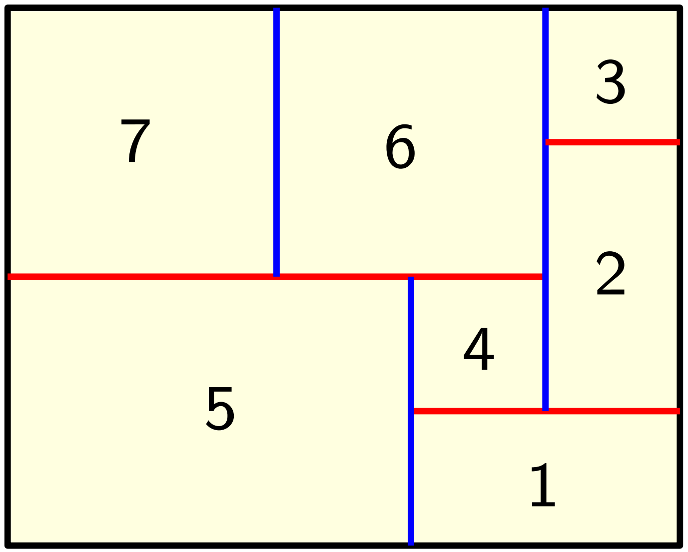
[Ackerman, Barequet, Pinter'06]

Contract top-left region:
two cases

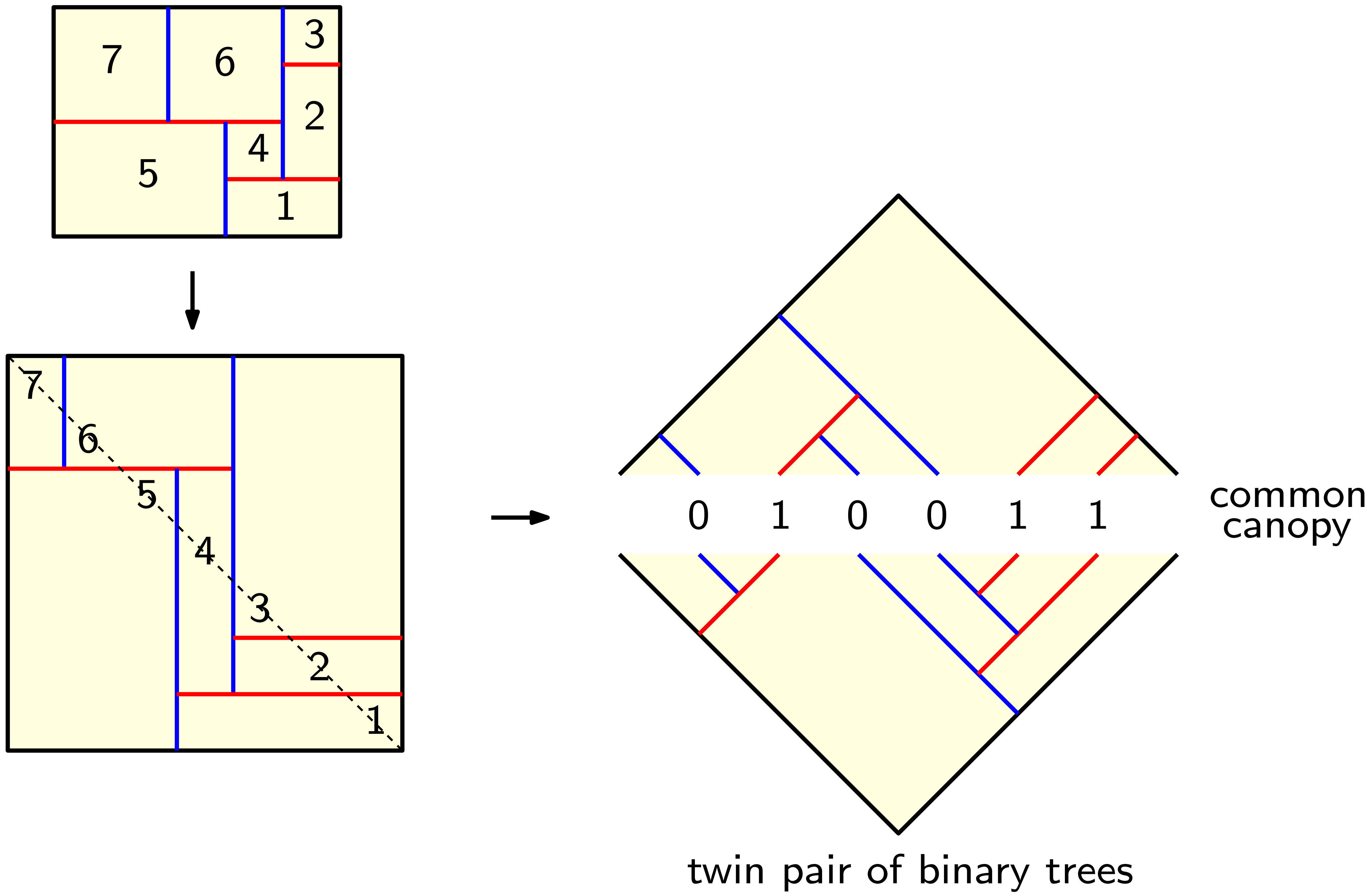




$\Rightarrow$ shelling order on regions

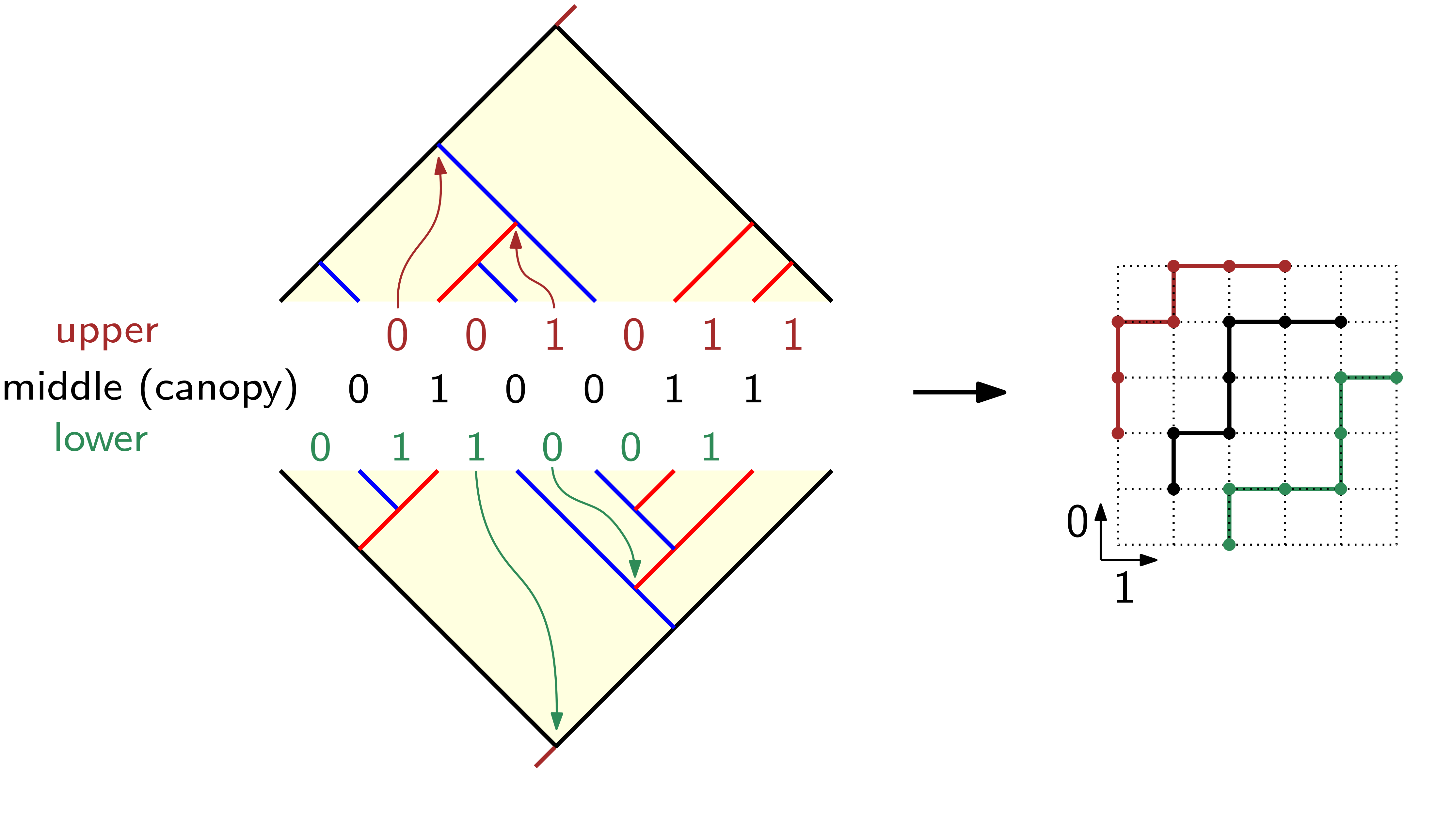


Diagonal representation

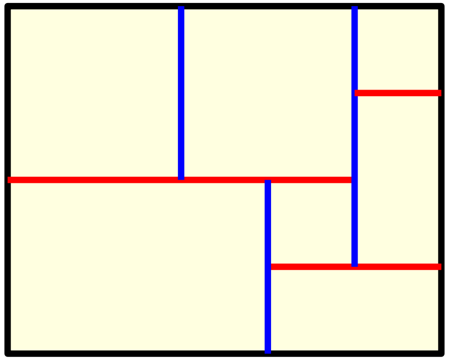


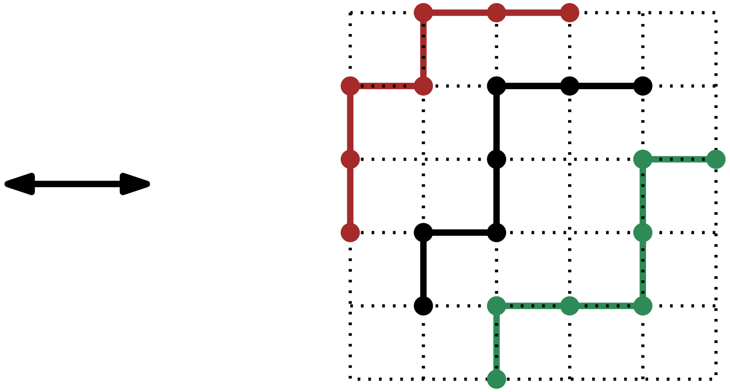
Encoding by a triple of walks

[Dulucq,Guibert'98]



Baxter numbers and Baxter families





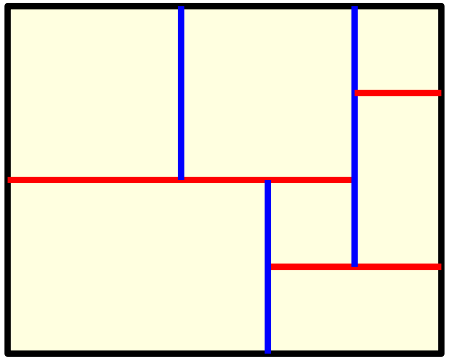
$$\text{Gessel-Viennot} \Rightarrow \boxed{w_n = \frac{2}{n(n+1)^2} \sum_{r=0}^{n-1} \binom{n+1}{r} \binom{n+1}{r+1} \binom{n+1}{r+2}}$$ Baxter numbers

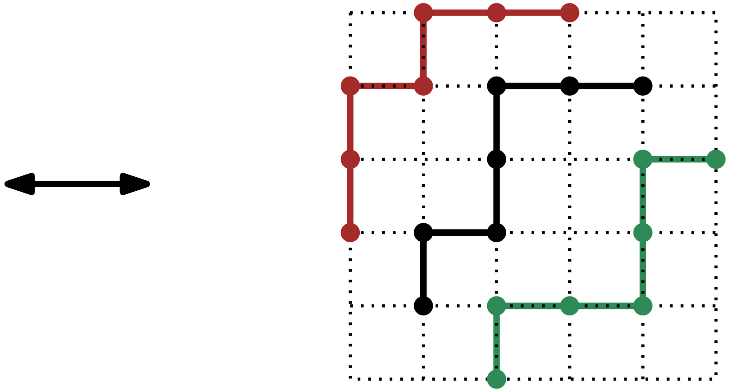
$$w_n \sim \frac{2^5}{\pi\sqrt{3}} 8^n n^{-4}$$

Baxter families are families counted by Baxter numbers

among which Baxter permutations, plane bipolar orientations, ...

Baxter numbers and Baxter families





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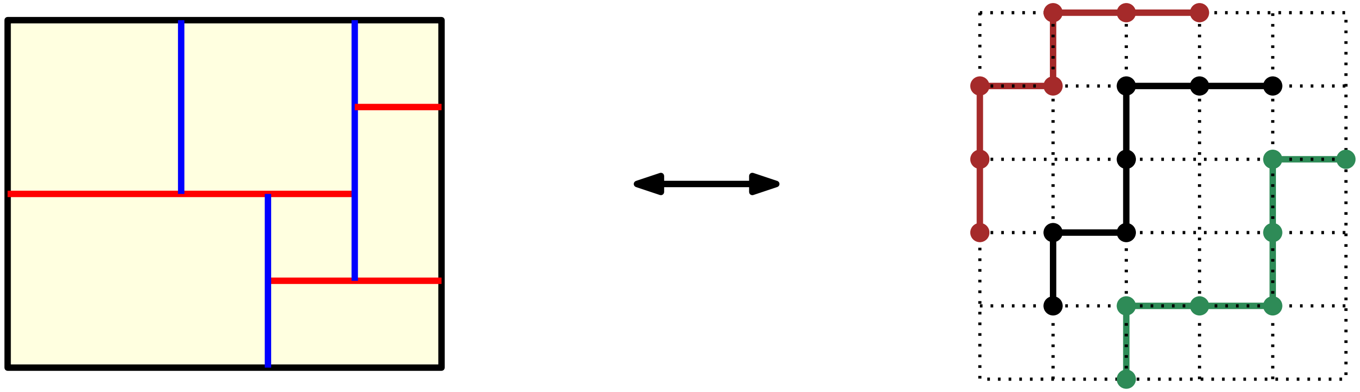

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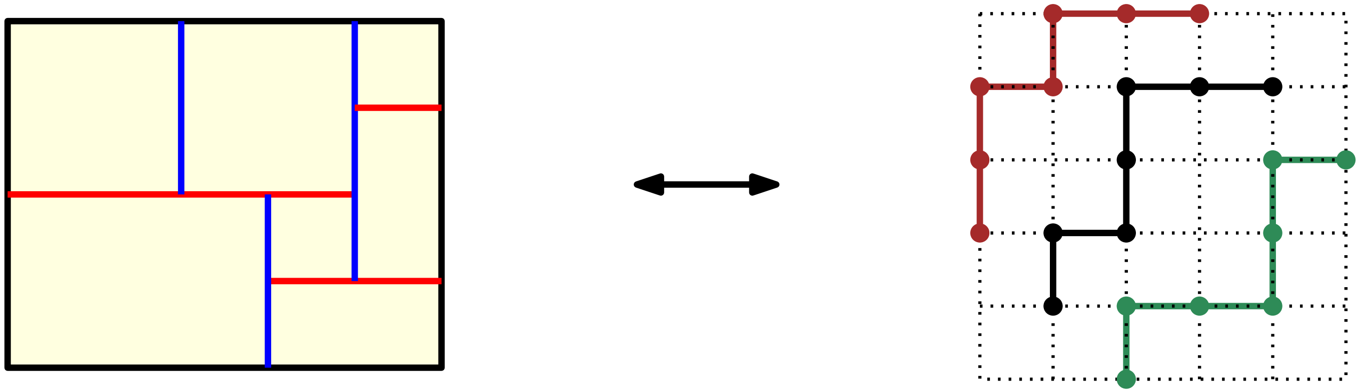
Various bijections relating these families (common generating tree)

[Viennot'81, Dulucq-Guibert'98, Ackerman-Barequet-Pinter'06, Felsner-F-Orden-Noy'11,...]

Baxter numbers and Baxter families



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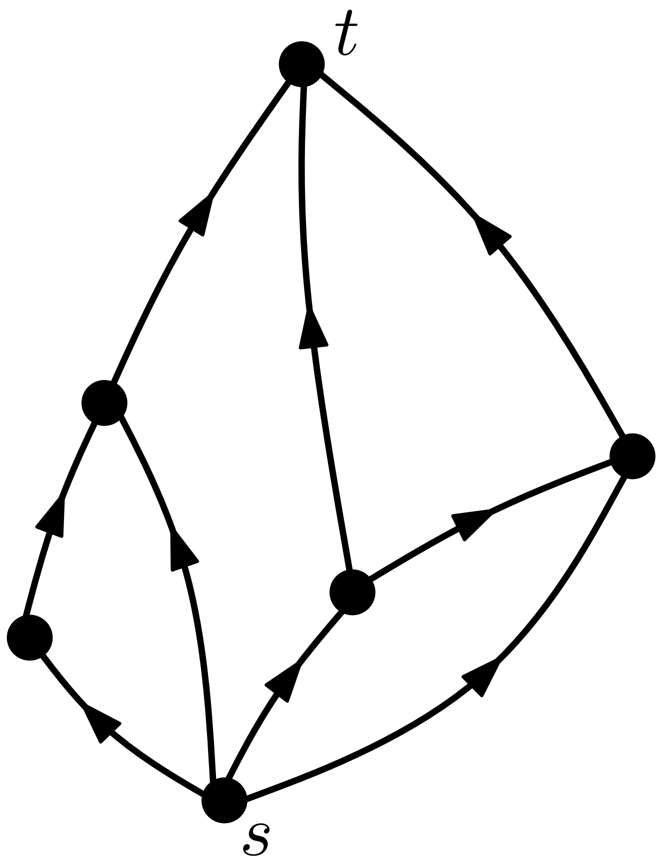
Link to weak order on permutations: [Reading'04,12]

mapping $\mathfrak{S}_n \to \mathcal{R}_n$

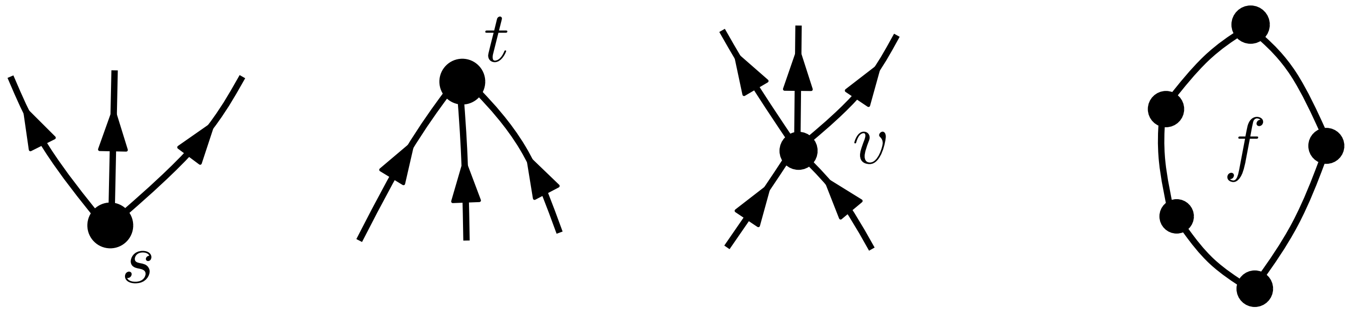
grouping permutations by rectangulation gives a lattice congruence

Plane bipolar orientations

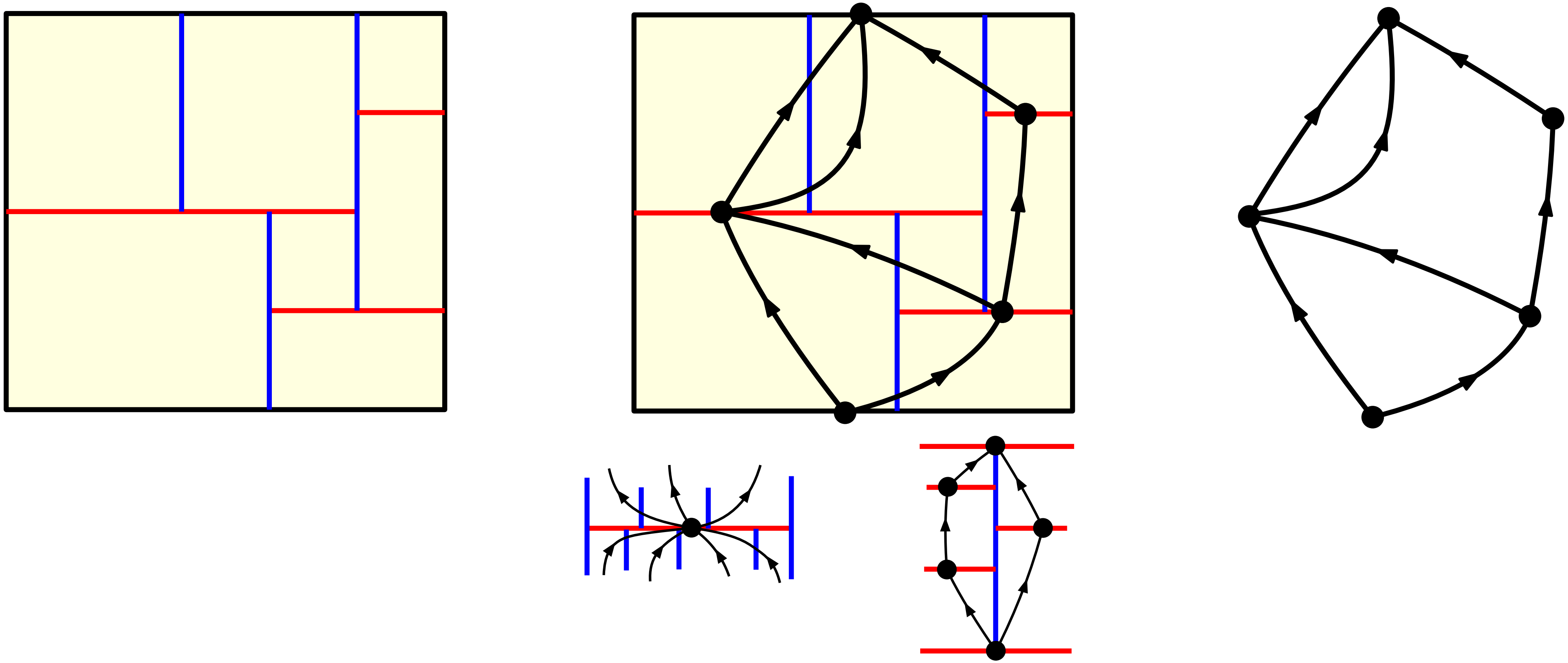


Acyclic orientation on planar map
with single min and single max
both incident to the outer face

Plane bipolar orientations ⇔ local conditions



Bijective link with weak rectangulations



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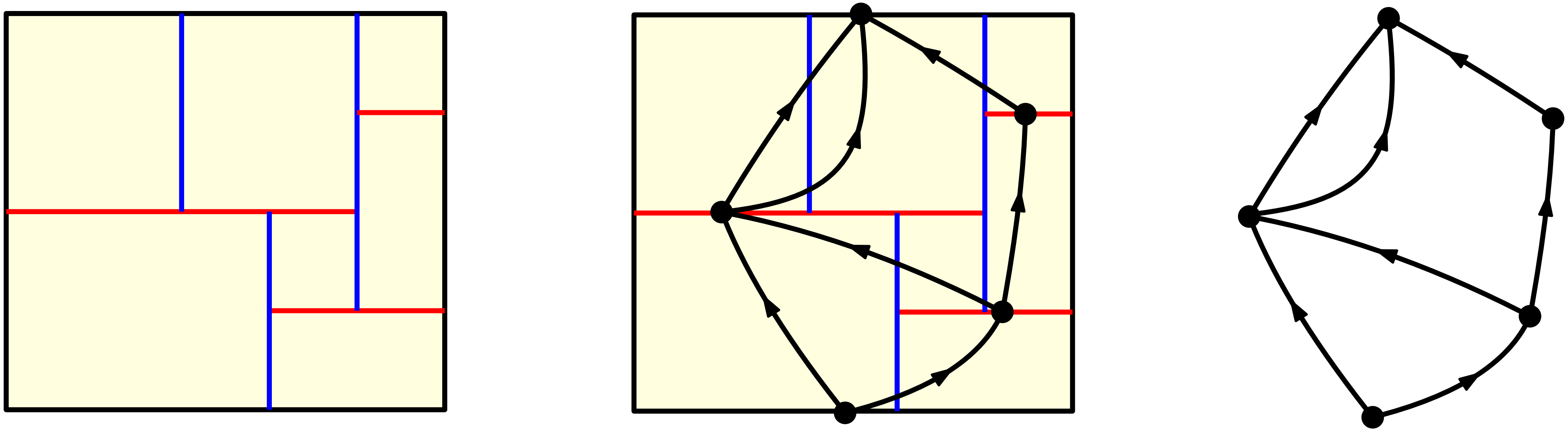
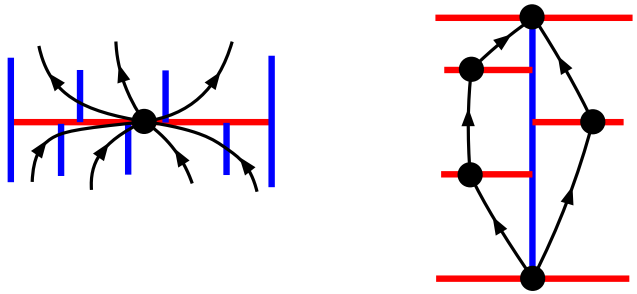
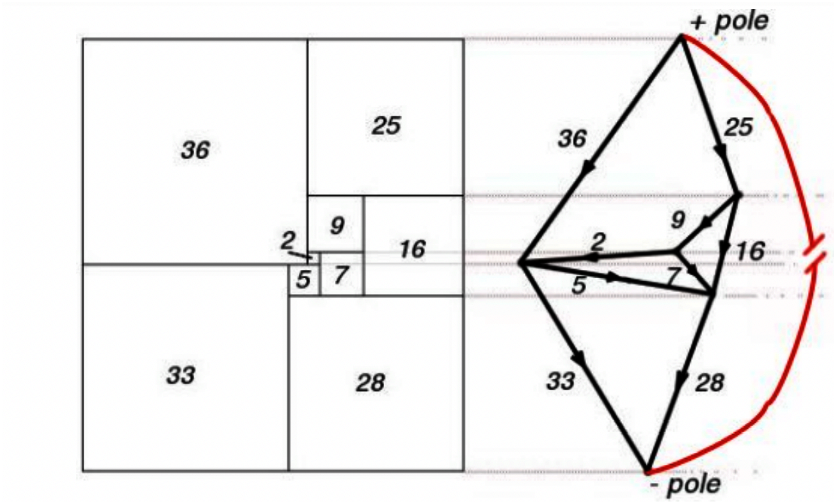




Correspondence used in problem "squaring the square"

[Brooks, Smith, Stone, Tutte'40]

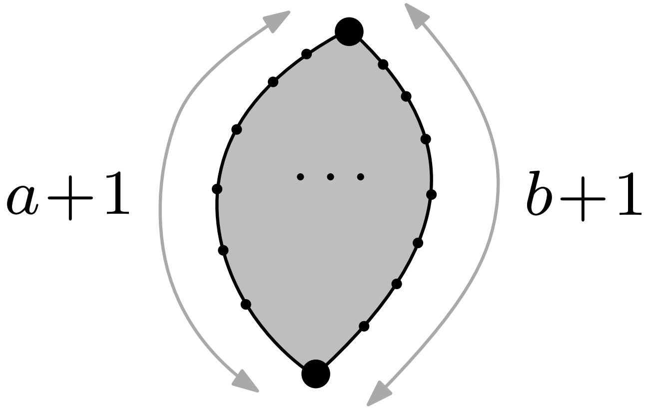


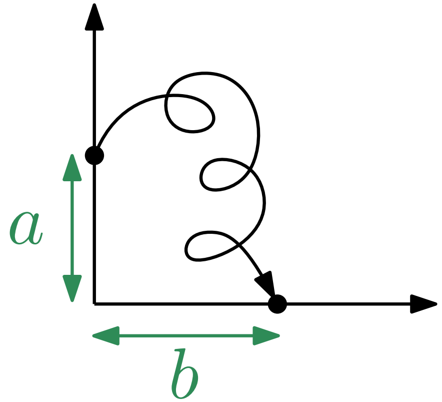
(www.squaring.net/history_theory/gfx/figure73.jpg)

Another walk-encoding: KMSW bijection

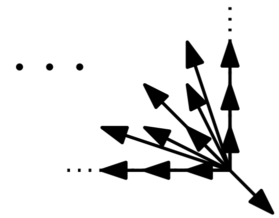
[Kenyon, Miller, Sheffield, Wilson'15]

Plane bipolar orientations ⟷ "Tandem walks" in the quadrant





step-set



SE $\cup \{(-i,j), i,j \geq 0\}$

n edges ⟷ length $n-1$

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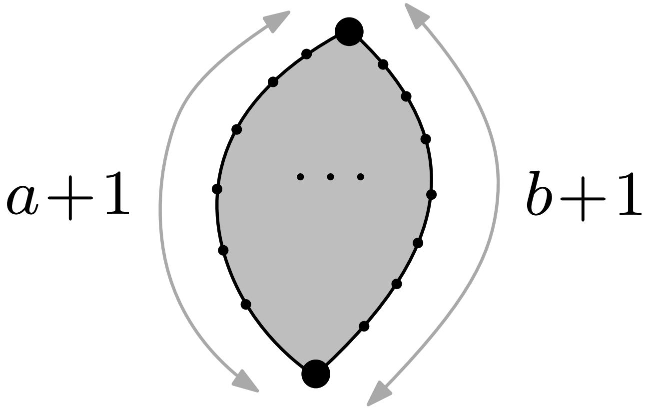
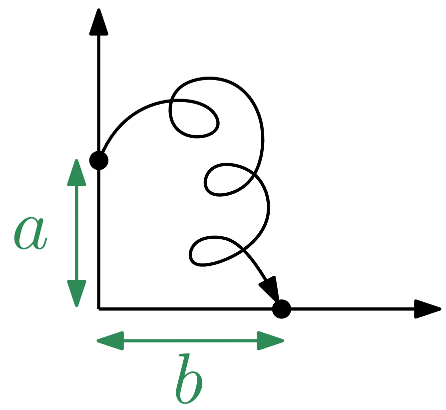
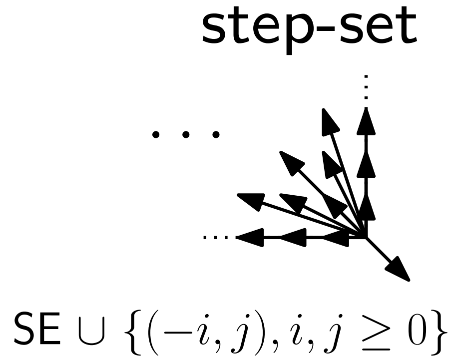
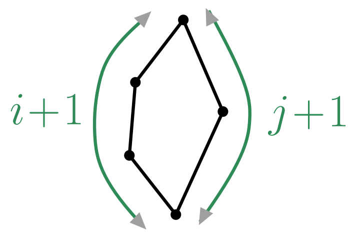






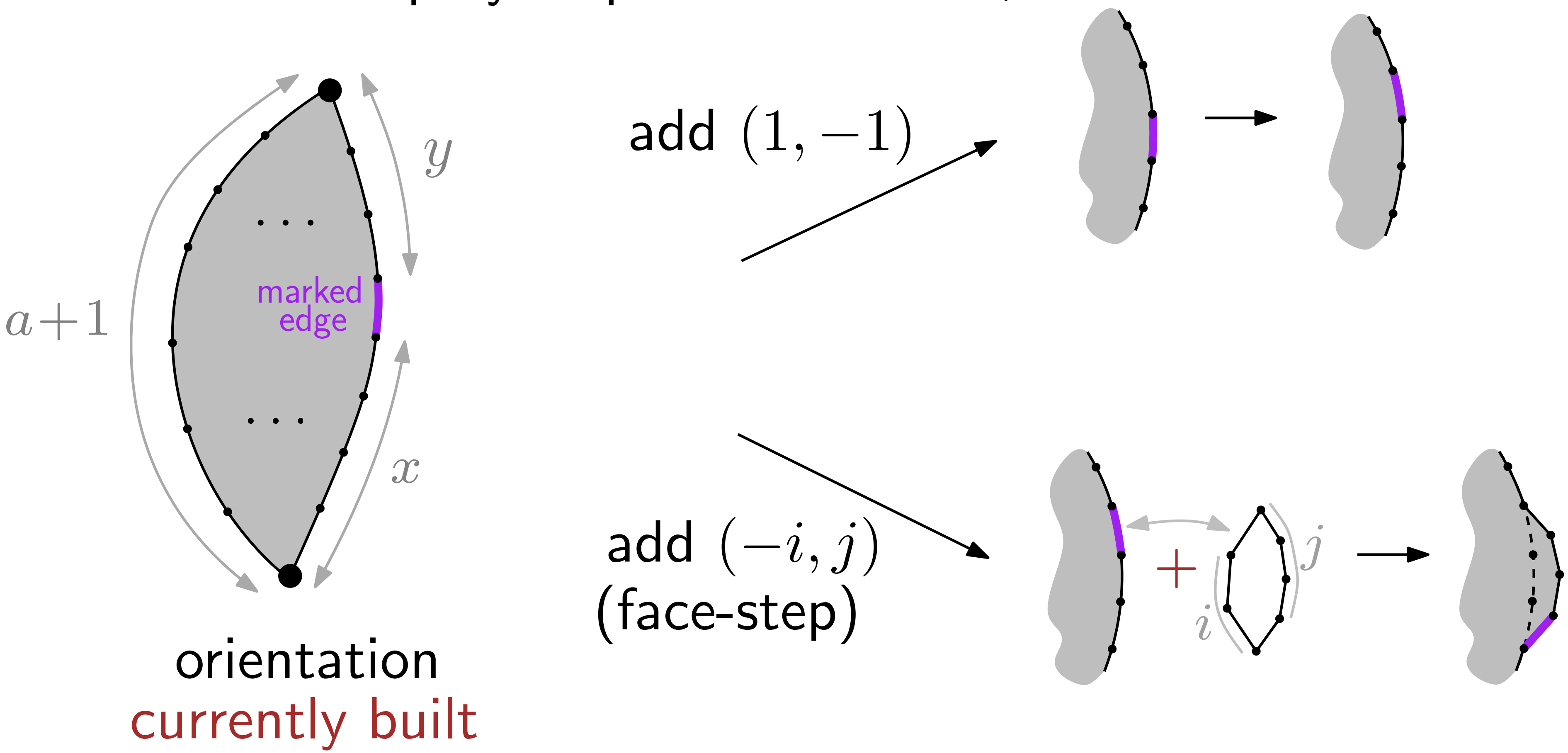
n edges ⟷ length $n-1$

face  ⟷ face-step $(-i,j)$

non-pole vertex ⟷ SE step

Another walk-encoding: KMSW bijection

Orientation is built step by step from the walk,



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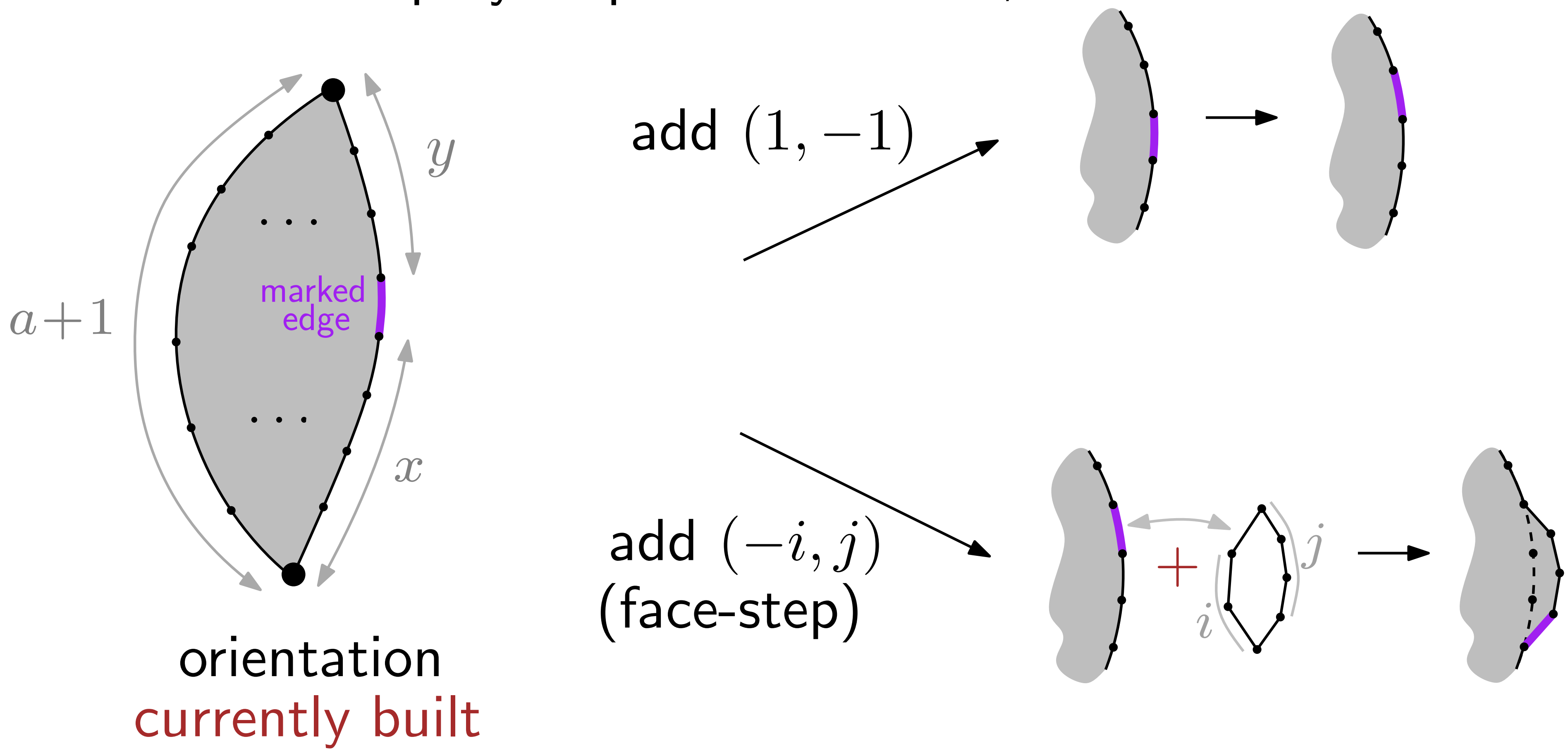
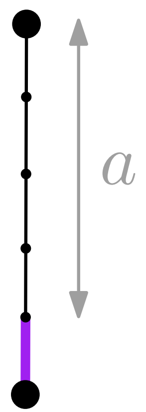
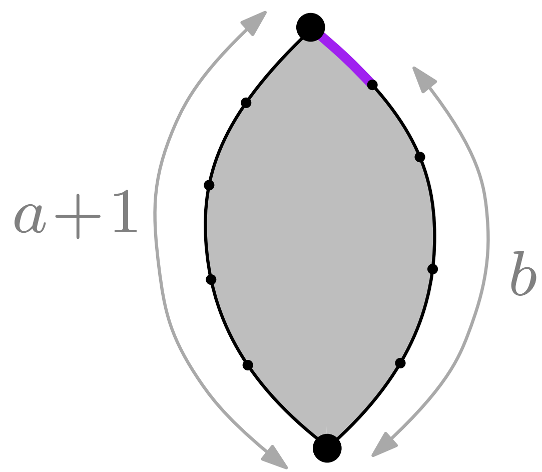


Starts with

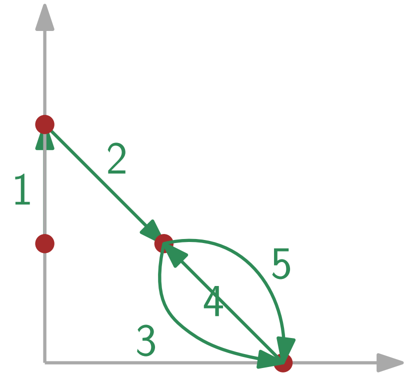


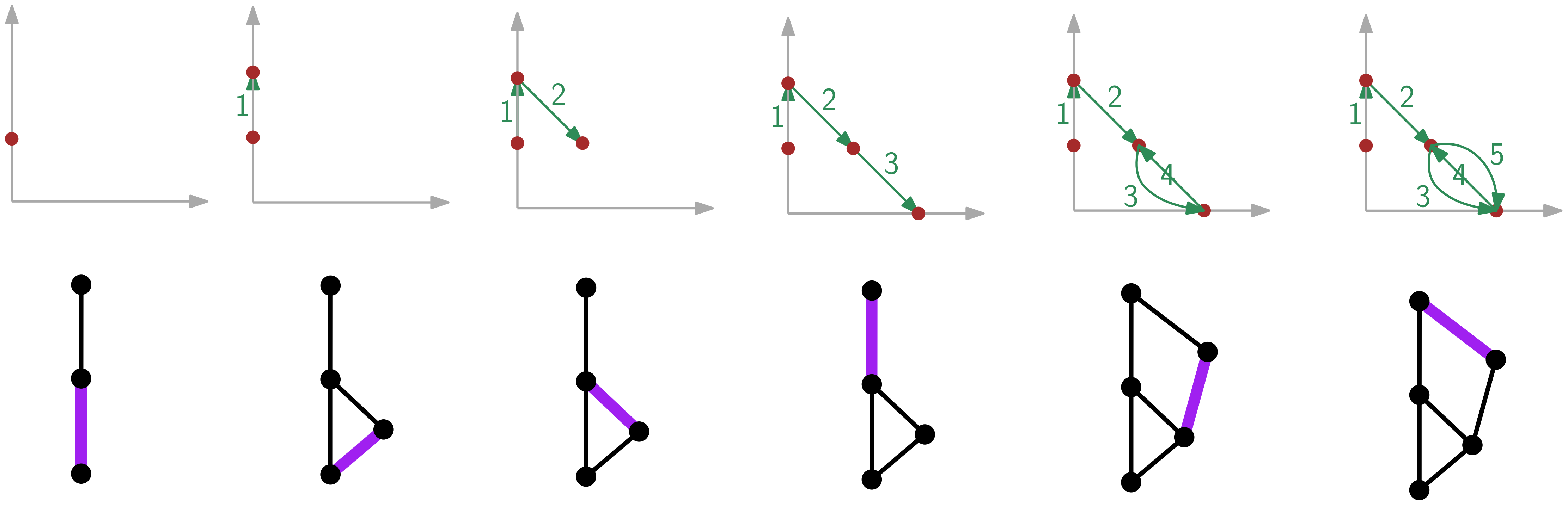
Ends with



Another walk-encoding: KMSW bijection

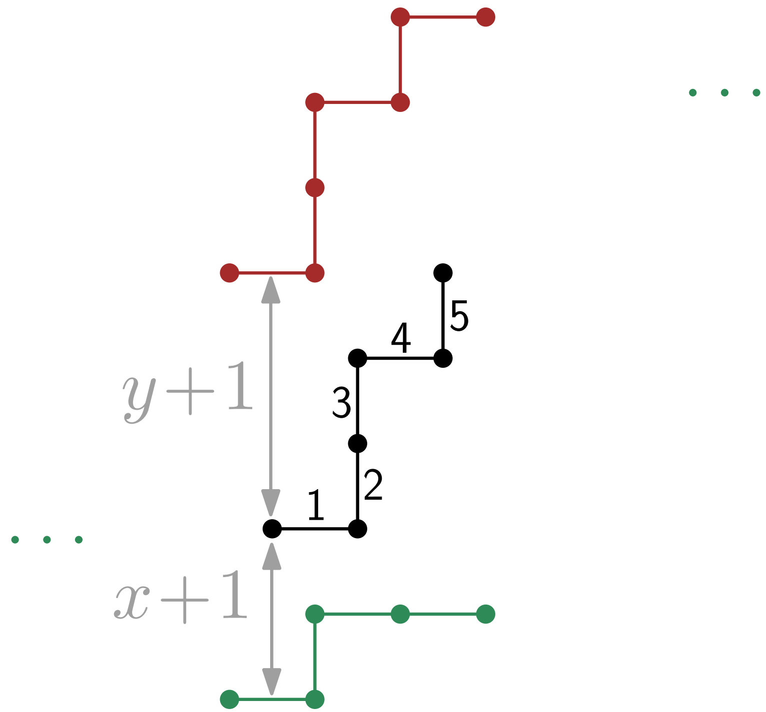
Example: build orientation associated to



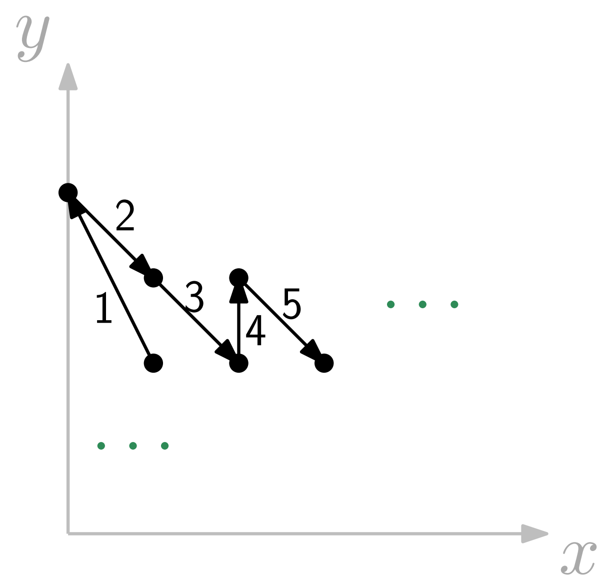


Link with non-intersecting triples of walks

[Bousquet-Mélou,F,Raschel'20]

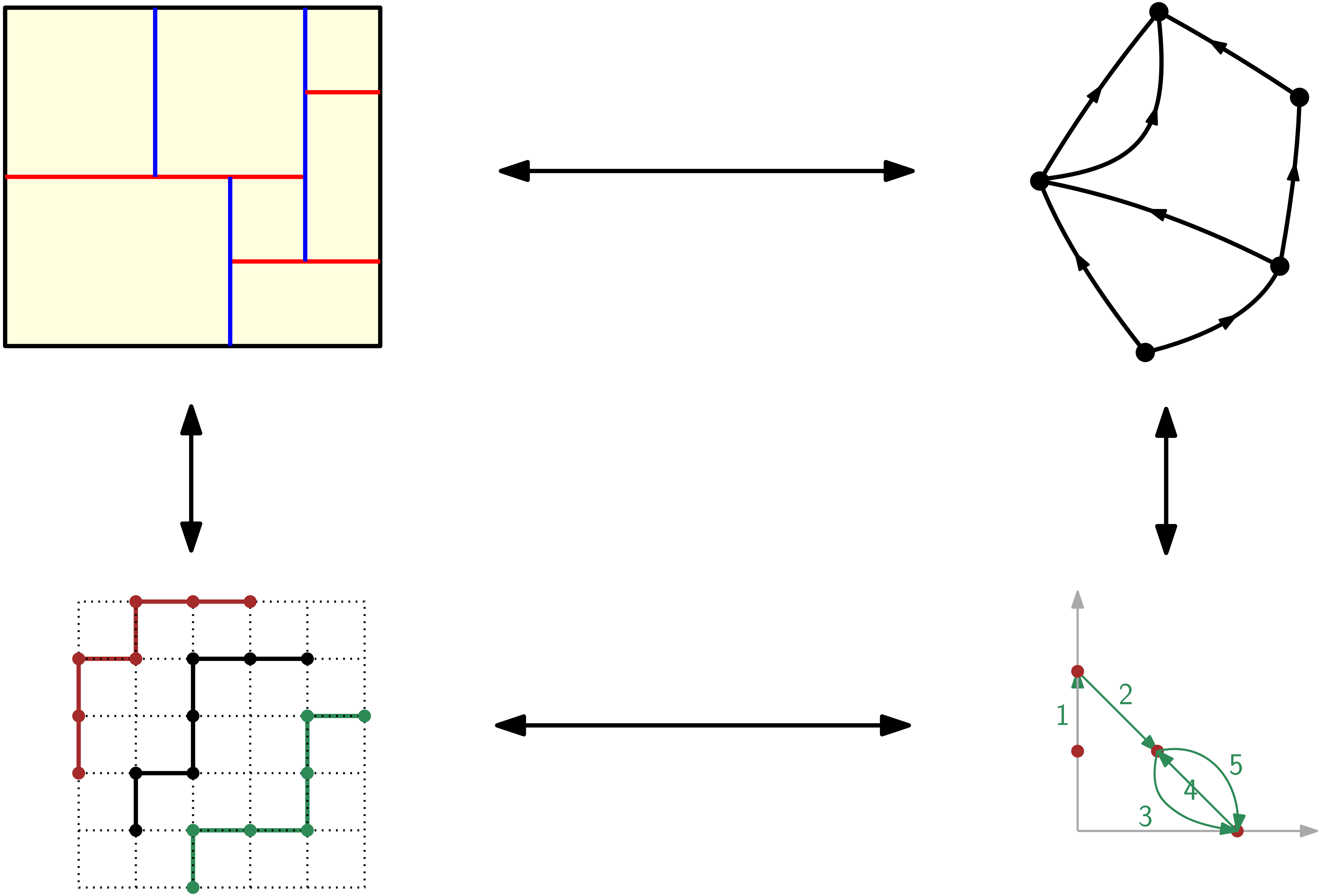


non-intersecting triple

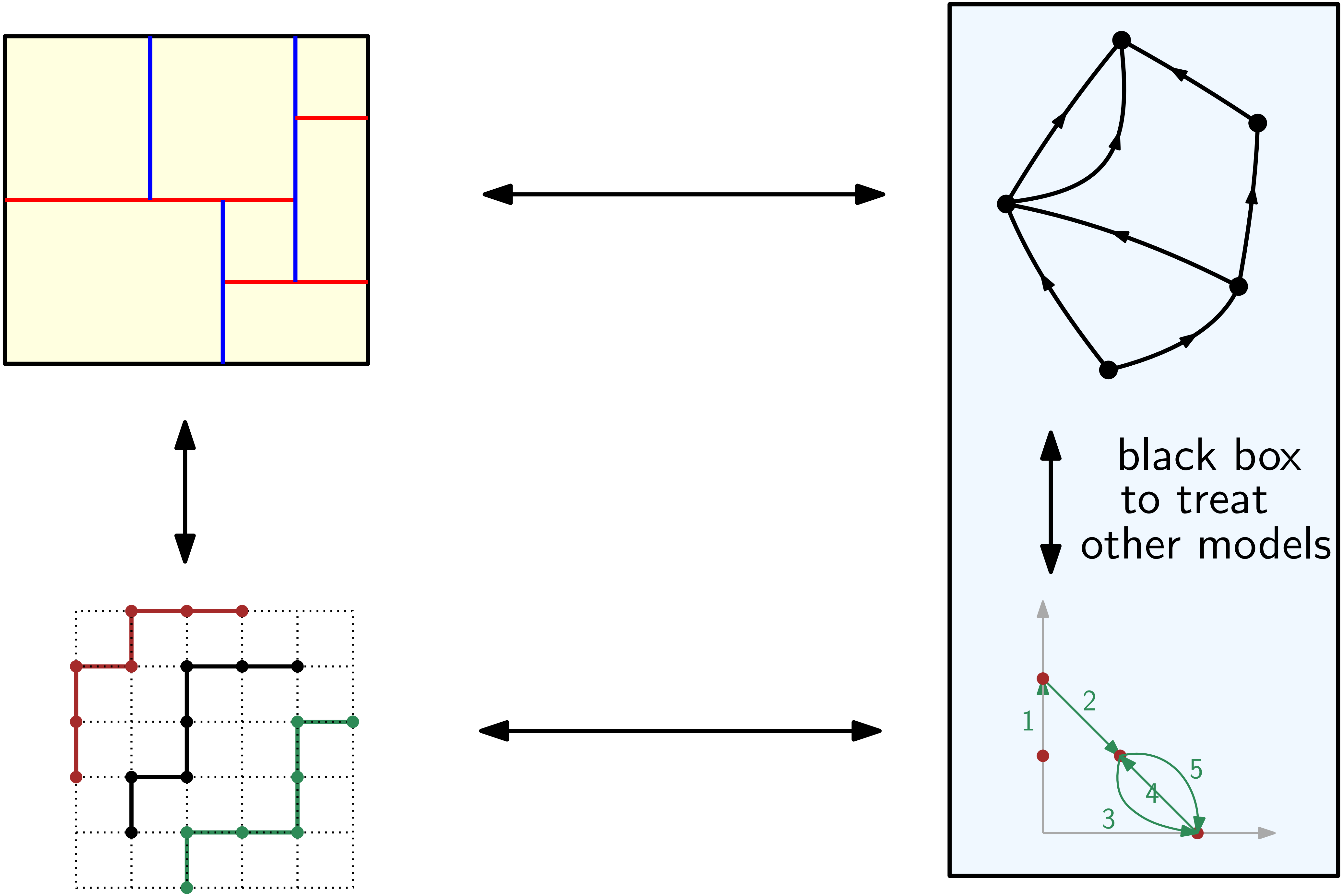


tandem walk

Summary of bijections so far

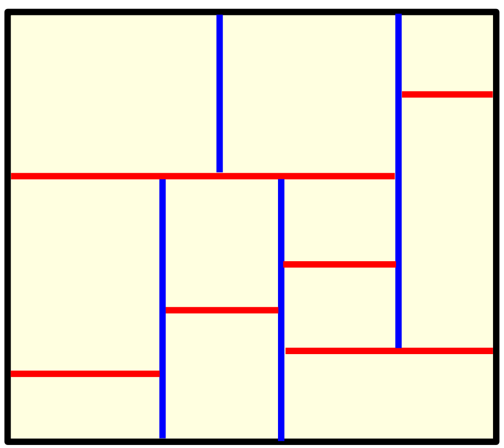


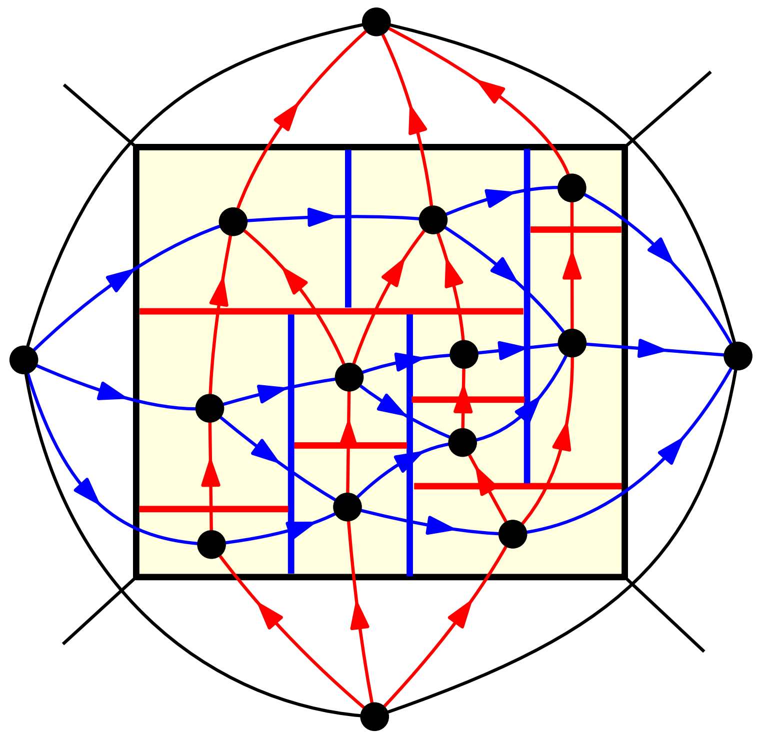
Summary of bijections so far

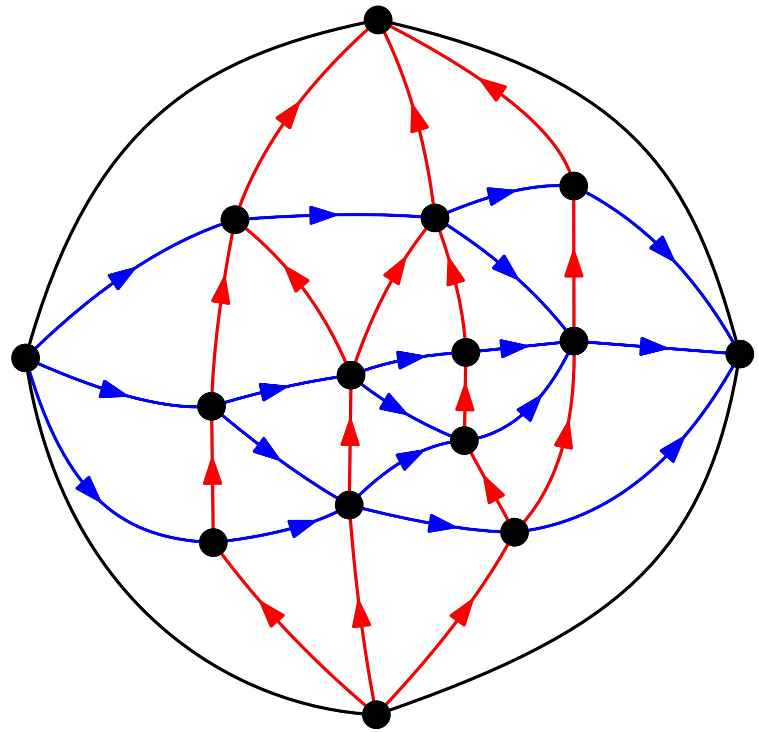


Strong rectangulations

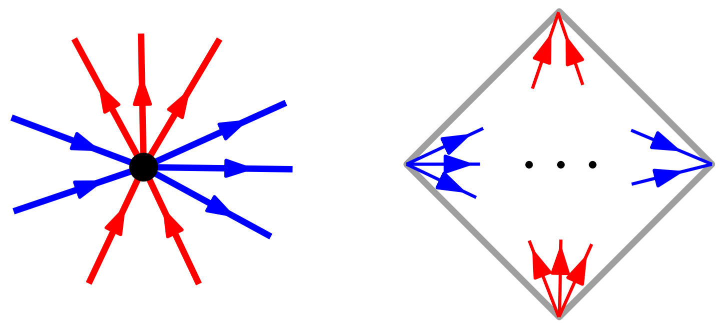
Model of decorated maps via duality [He'93]





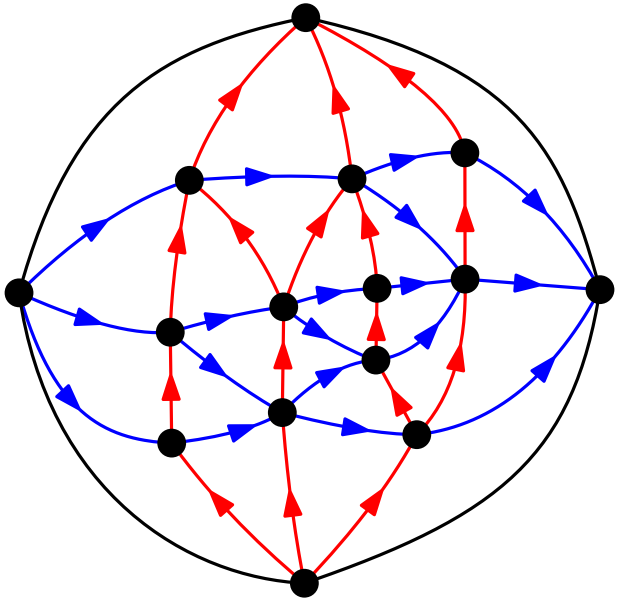


Pair of transversal plane bipolar orientations



Local conditions

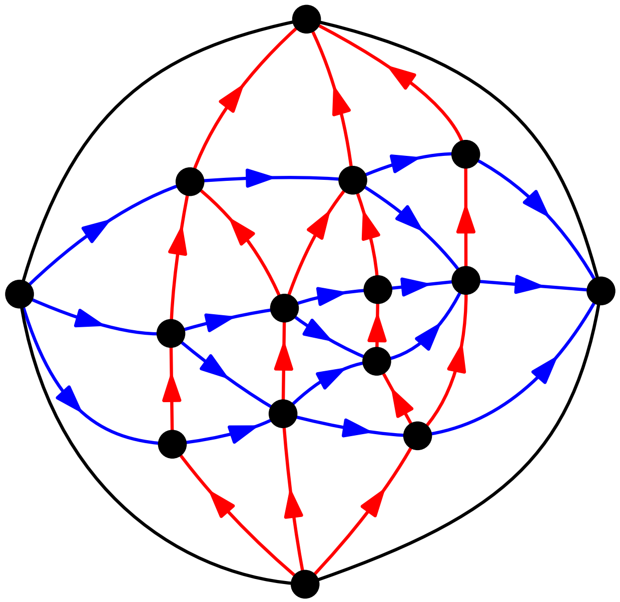
Encoding by (weighted) tandem walks



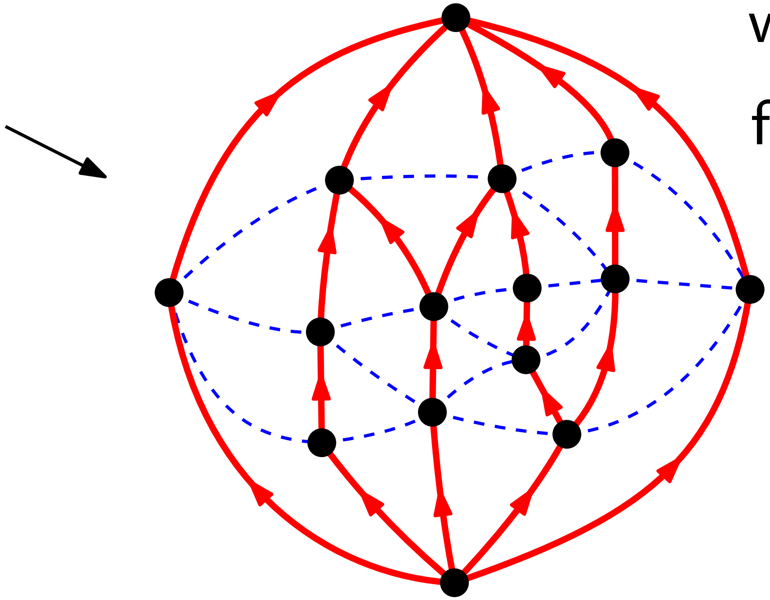
Transversal structure
$n + 4$ vertices

Encoding by (weighted) tandem walks

[F-Narmanli-Schaeffer'21]



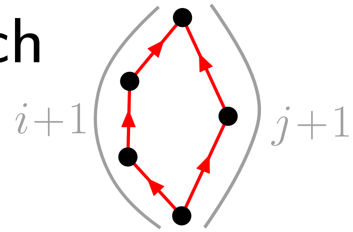
Transversal structure
$n+4$ vertices



red bipolar poset
$+$ transversal edges

weight $\binom{i+j-2}{i-1}$

for each



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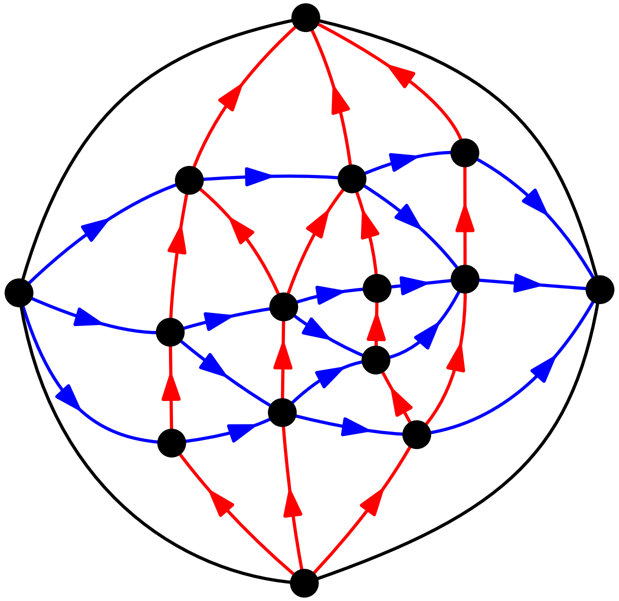

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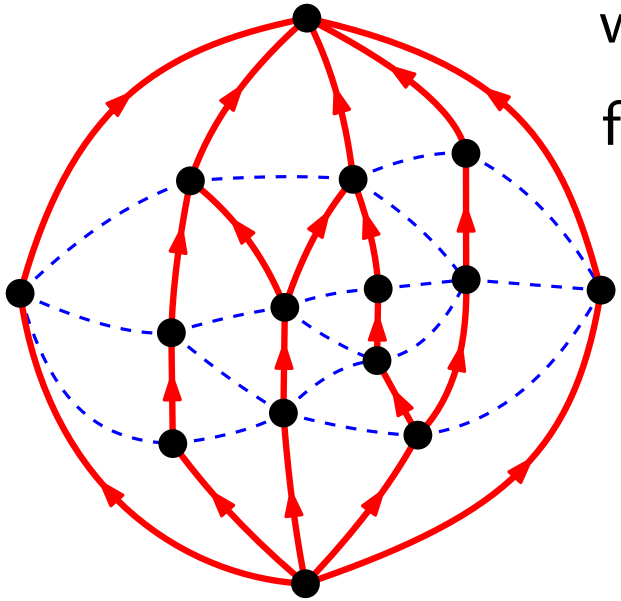

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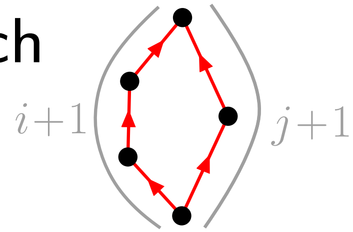
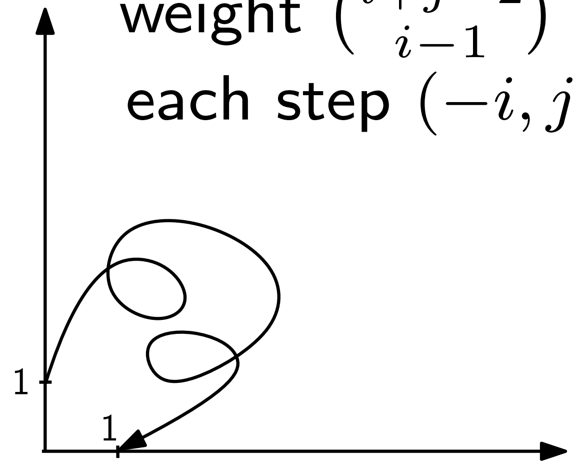
for each



weight $\binom{i+j-2}{i-1}$ for each step $(-i, j)$

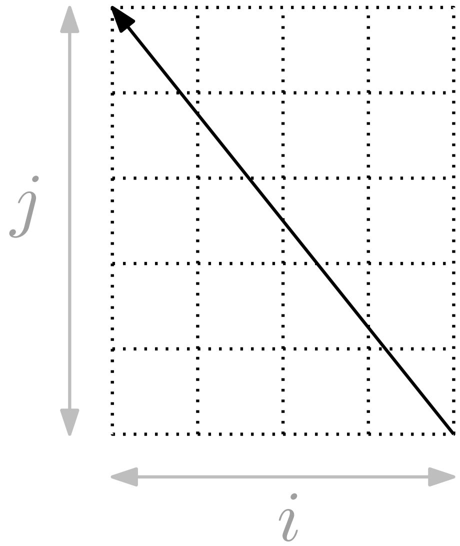


weighted tandem walk
with n SE steps

Encoding by tandem walks with small steps

[F-Narmanli-Schaeffer'21]

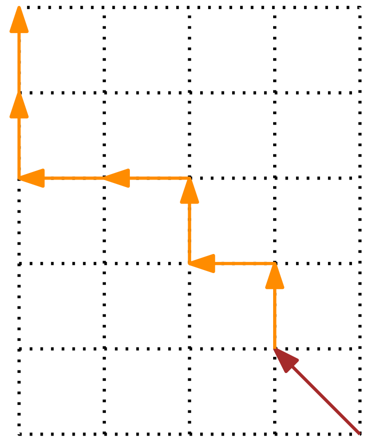
face-step



weight $\binom{i+j-2}{i-1}$

$\Leftrightarrow$

small-step portion



$\Rightarrow s_n = \#$ quadrant walks with steps in $\{SE, N, W, NW\}$

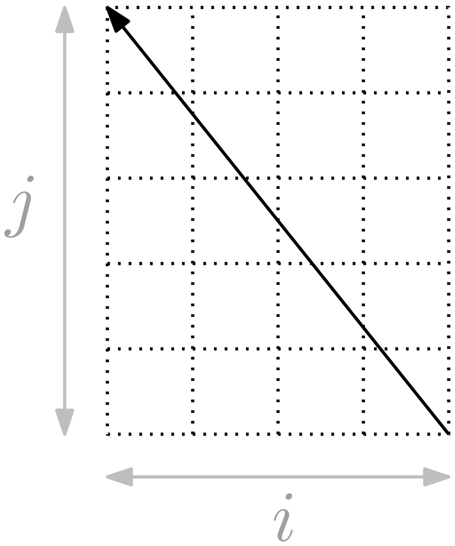
from $(0,1)$ to $(1,0)$, with $n-2$ SE steps

steps SE can not be followed by N or W

Encoding by tandem walks with small steps

[F-Narmanli-Schaeffer'21]

face-step



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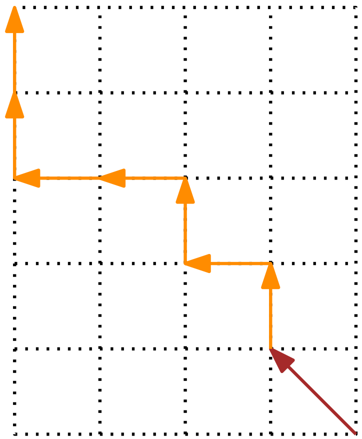
$\Leftrightarrow$

small-step portion



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from $(0,1)$ to $(1,0)$, with $n-2$ SE steps

steps SE can not be followed by N or W

$\Rightarrow$ explicit recurrence

1, 2, 6, 24, 116, 642, 3938, 26194, 186042 (A342141 in OEIS)

other recurrence (& small step walks) [Inoue, Takahashi, Fujimaki'09]

Asymptotic enumeration

[F-Narmanli-Schaeffer'21]

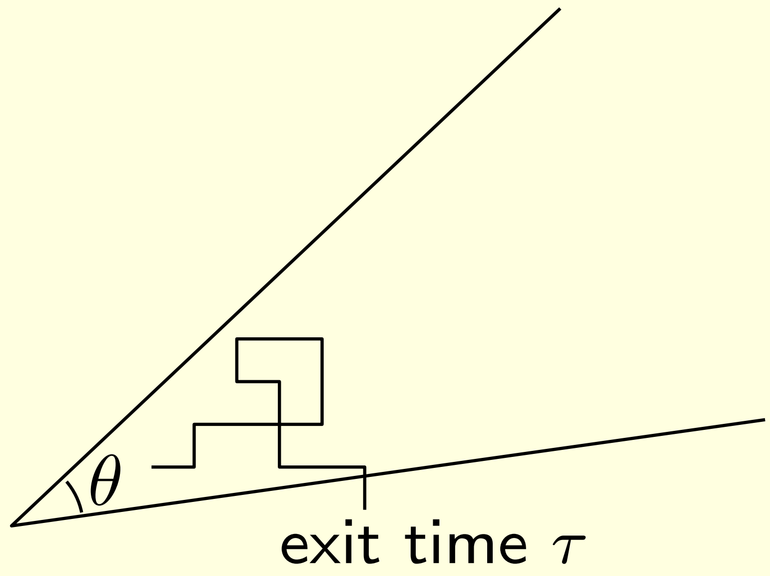
relies on [Denisov-Wachtel'11, Bostan-Raschel-Salvy'14]

Each of the counting sequences w_n, s_n has asymptotics of the form

$$c\,\gamma^n\,n^{-\alpha}$$

$\alpha = 1 + \frac{\pi}{\theta}$

	weak	strong
γ	8	$27/2$
$\cos(\theta)$	$1/2$	$7/8$
α	4	$\approx 7.21 \notin \mathbb{Q}$



$$\mathbb{P}(\tau > n) \sim c\,n^{-\frac{\pi}{2\theta}}$$

$$\mathbb{P}(\tau > n \;\&\; \text{excursion}) \sim c'\,n^{-1-\frac{\pi}{\theta}}$$

Asymptotic enumeration

[F-Narmanli-Schaeffer'21]

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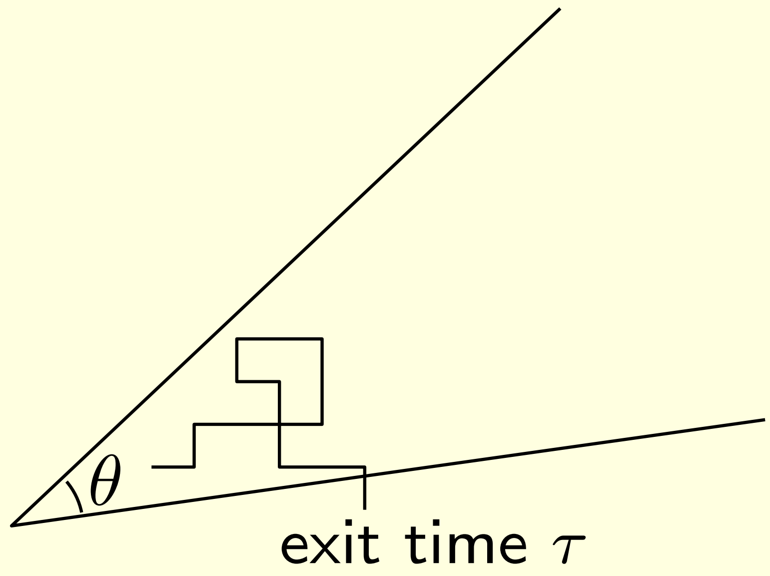
$\notin \mathbb{Q}$ → not D-finite



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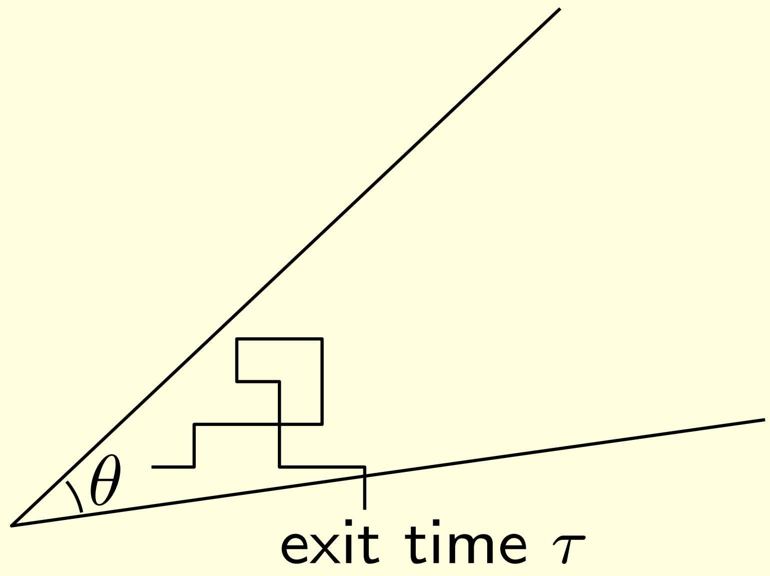

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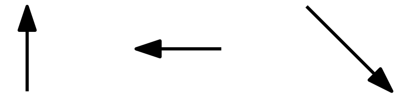
$\mathbb{P}(\tau > n \,\&\, \text{excursion}) \sim c'\,n^{-1-\frac{\pi}{\theta}}$

	weak	strong
γ	8	$27/2$
$\cos(\theta)$	$1/2$	$7/8$
α	4	$\approx 7.21 \notin \mathbb{Q}$

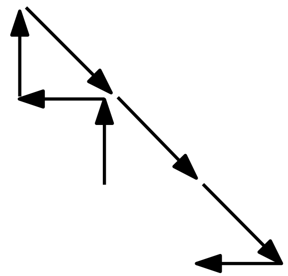
$27/2$ → optimal encoding [Takahashi, Fujimaki, Inoue'09] $s_n \leq \binom{3n}{n} 2^n$

$\approx 7.21 \notin \mathbb{Q}$ → not D-finite

Illustration on tandem walks with small steps

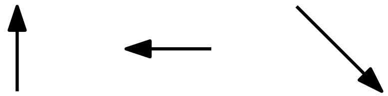
Step-set 
N W SE

(triangulated bipolar orientations)

Random walk 

$\mathbb{P}(\text{each step}) = \frac{1}{3}$

Illustration on tandem walks with small steps

Step-set 

N W SE

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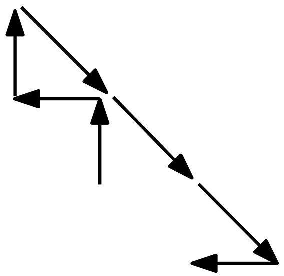
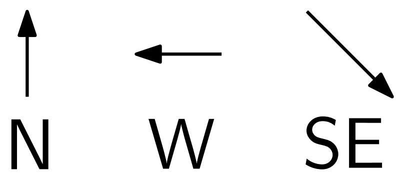
Random walk 

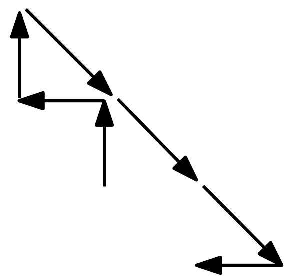
$\mathbb{P}(\text{each step}) = \frac{1}{3}$

$$\mathrm{Cov} = \begin{pmatrix} \mathbb{E}(X^2) & \mathbb{E}(XY) \\ \mathbb{E}(XY) & \mathbb{E}(Y^2) \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

Illustration on tandem walks with small steps

Step-set 

(triangulated bipolar orientations)

Random walk 

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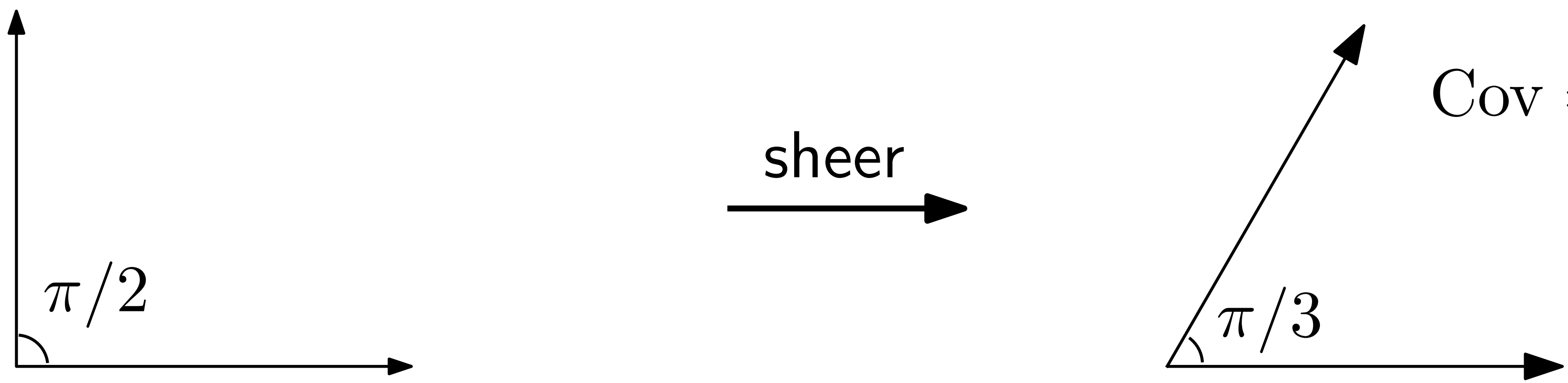


$$\mathrm{Cov} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

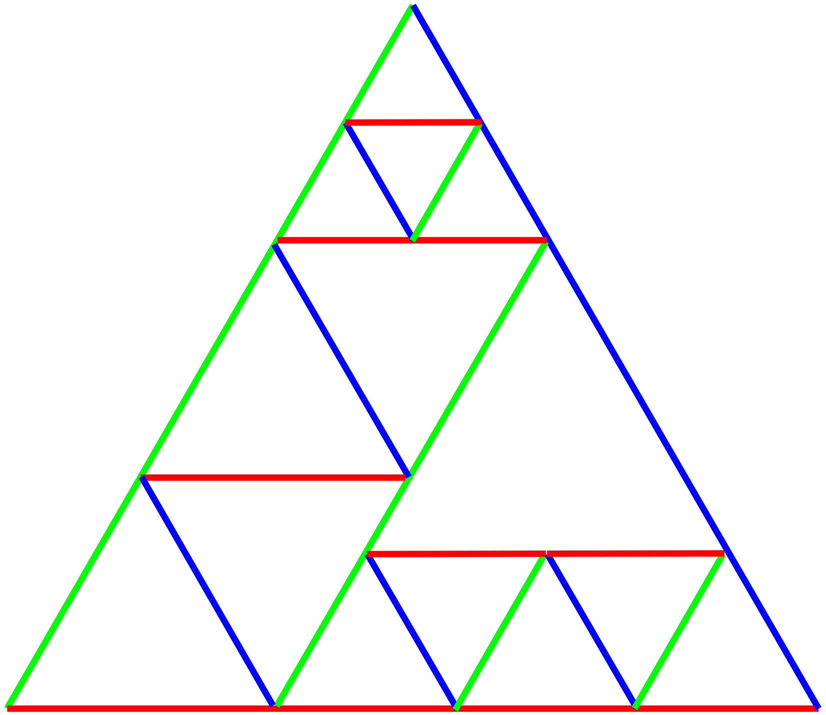
$\Rightarrow$ # quadrant excursions length $3n \sim c \cdot 27^n n^{-4}$

($\alpha = 4$ universal for plane bipolar orientations)

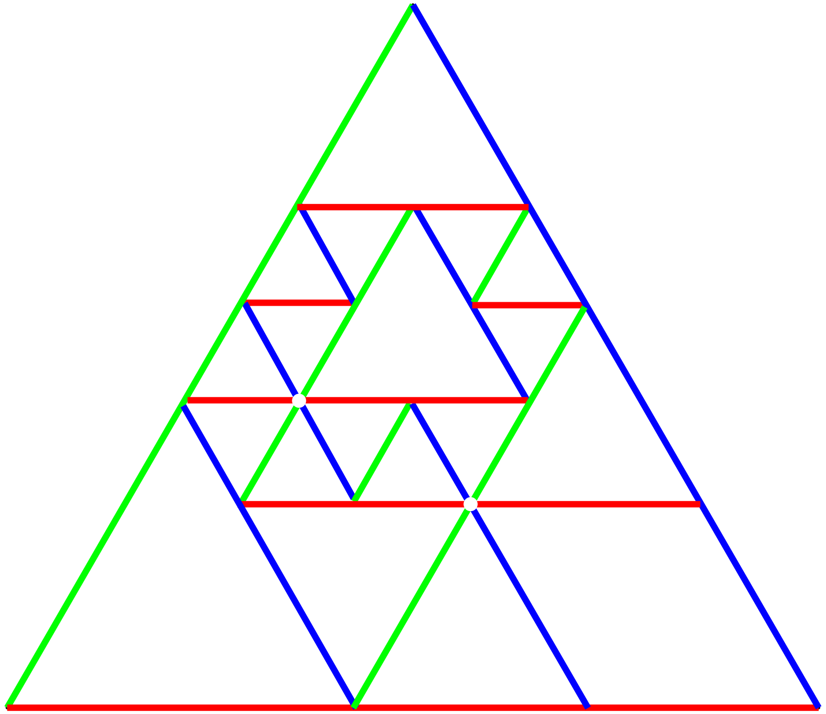
Corner polyhedra

(tricolored contact-systems)

Tricolored contact-systems

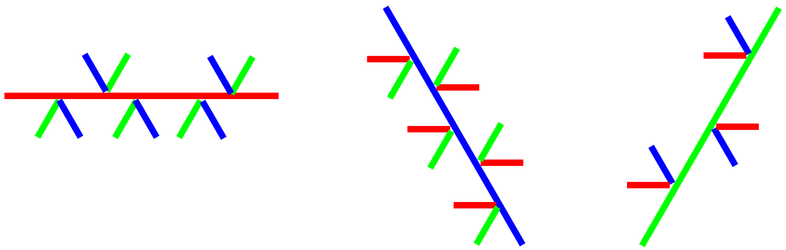


generic



Not generic

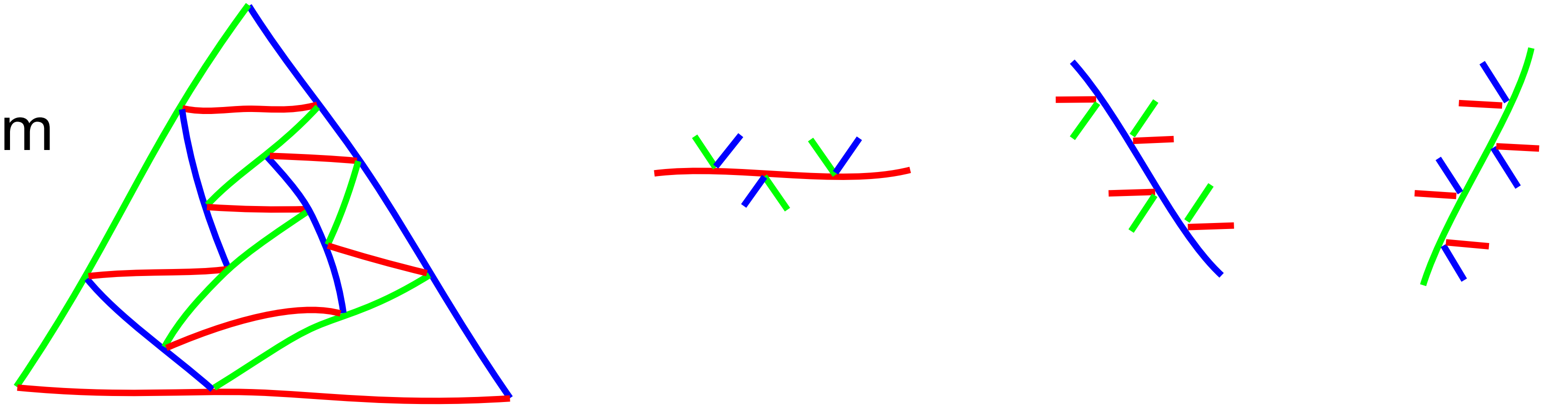
[Gonçalves'19]

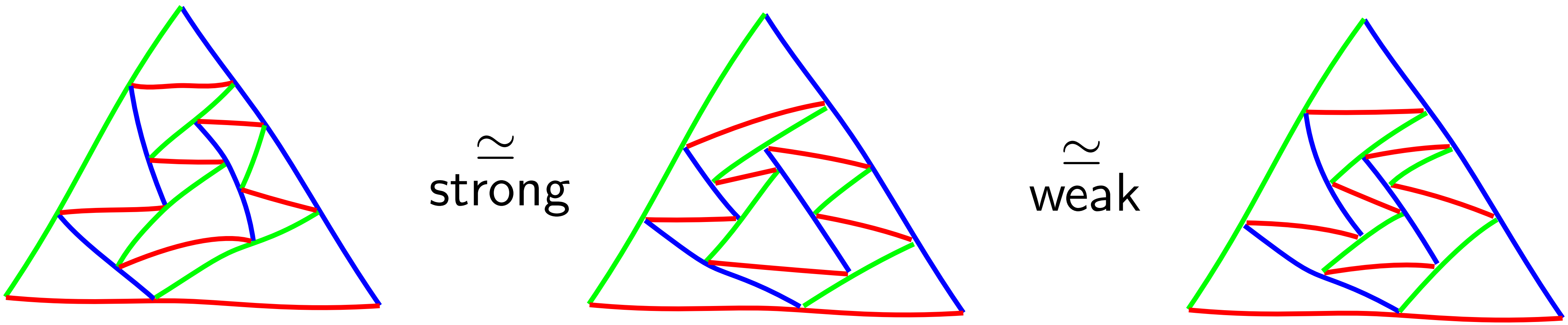


Rk: Very rigid (regions are equilateral triangles)

Relaxed tricolored contact-systems

Contact-system of curves



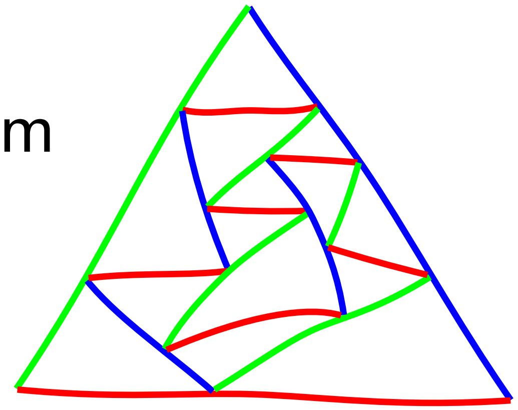


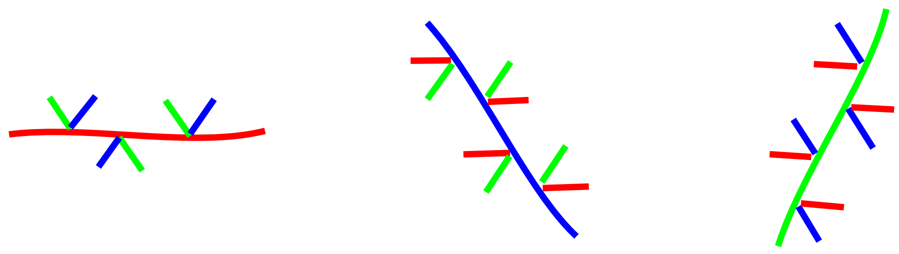
$w'_n =$ # weak equivalence classes with $2n$ regions

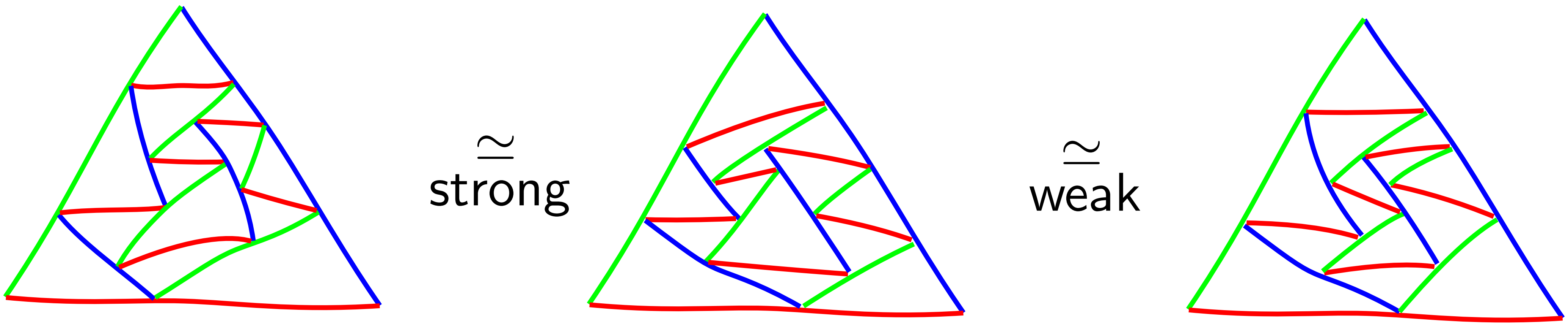
$s'_n =$ # strong equivalence classes with $2n$ regions

Relaxed tricolored contact-systems

Contact-system
of curves







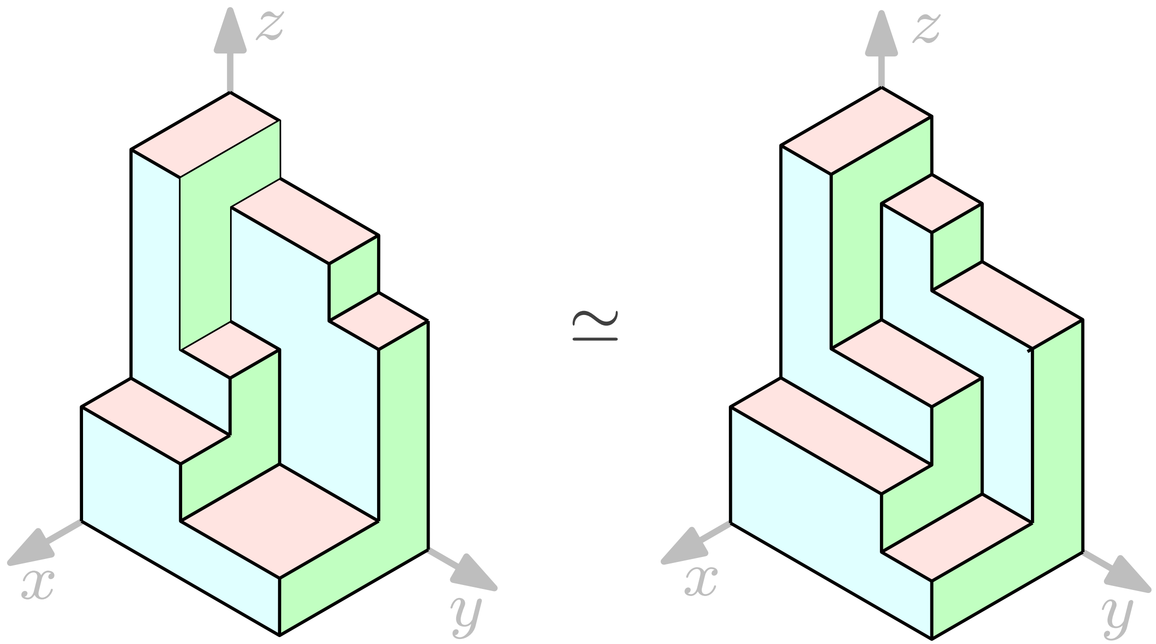
$w'_n = $ # weak equivalence classes with $2n$ regions
$s'_n = $ # strong equivalence classes with $2n$ regions

Rk: For bicolored systems, same equivalence classes in the relaxed version

Rectilinear representation: corner polyhedra

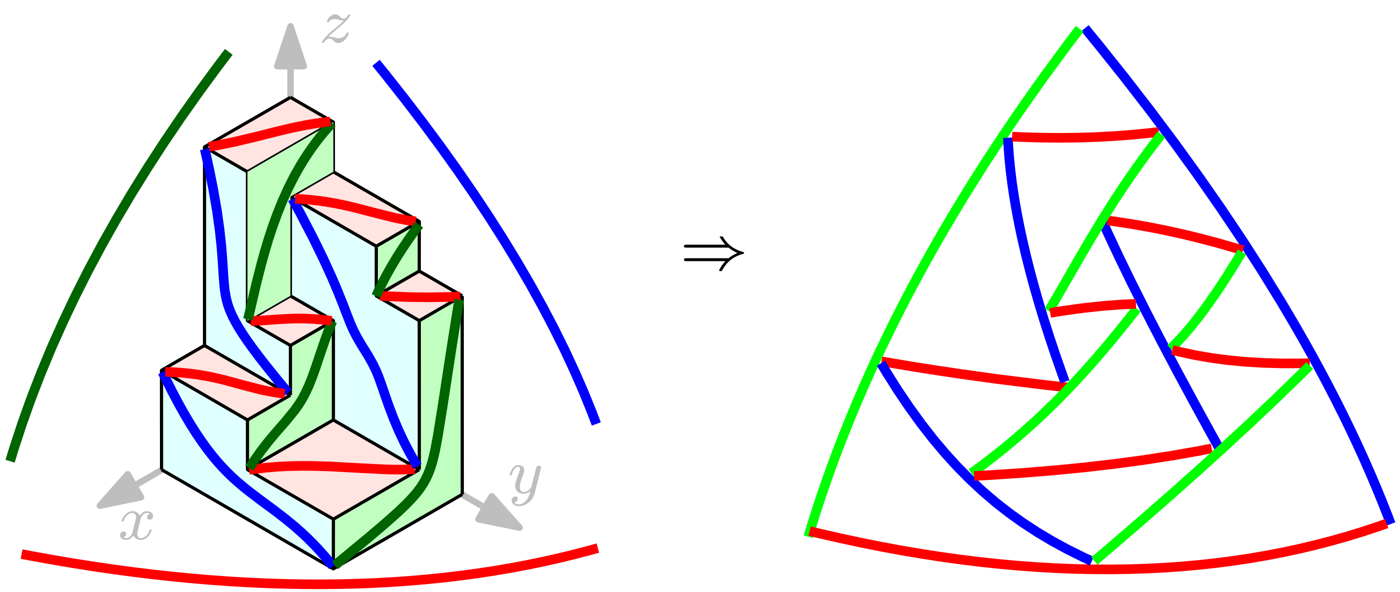
[Eppstein-Mumford'09]

3d-shape whose boundary is made of axis-orthogonal "flats"
at most 3 flats meet at any point, 3 of them point backward



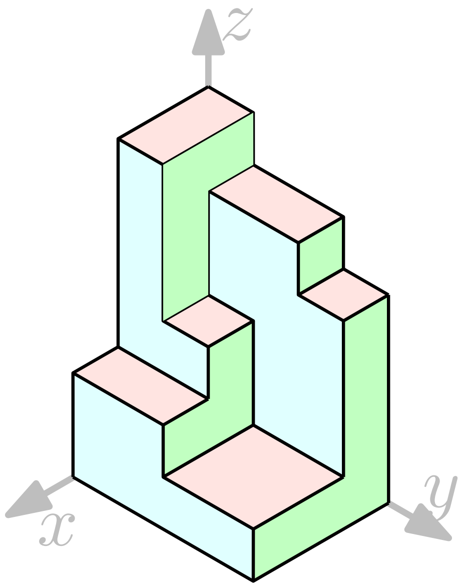
size = # flats - 3

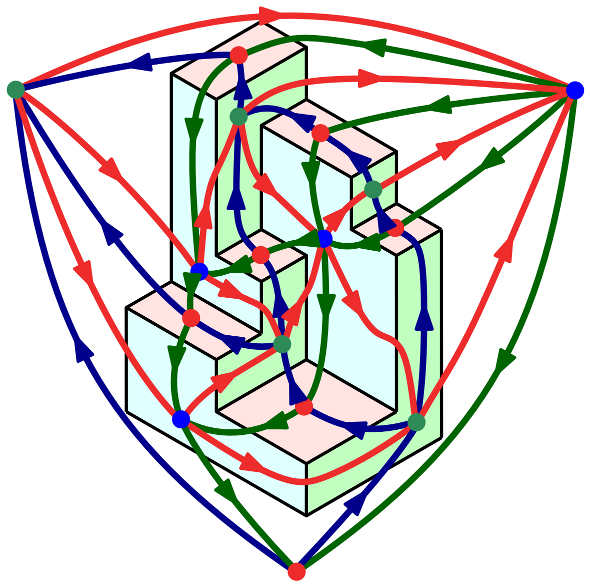
Bijection to weak contact-systems:

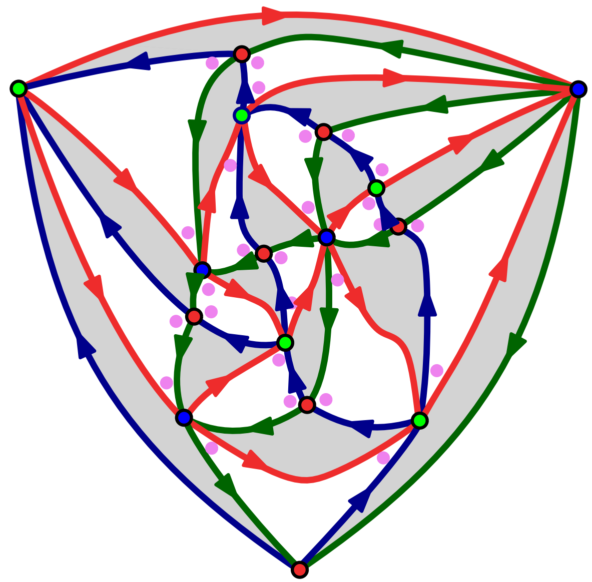


Decorated map and bipolar orientation

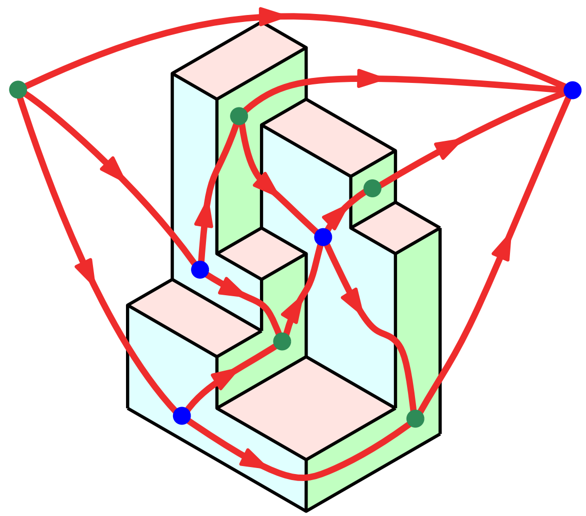
[Eppstein-Mumford'09]

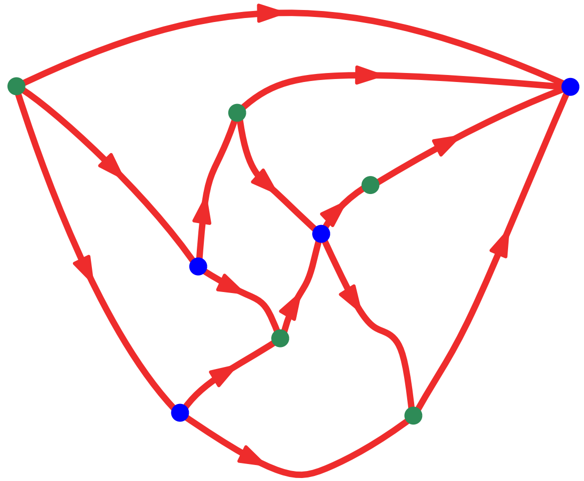






polyhedral orientation



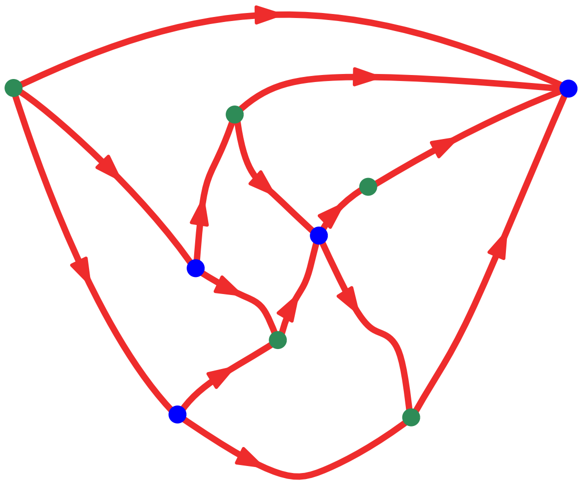


[F,Narmanli,Schaeffer'22]

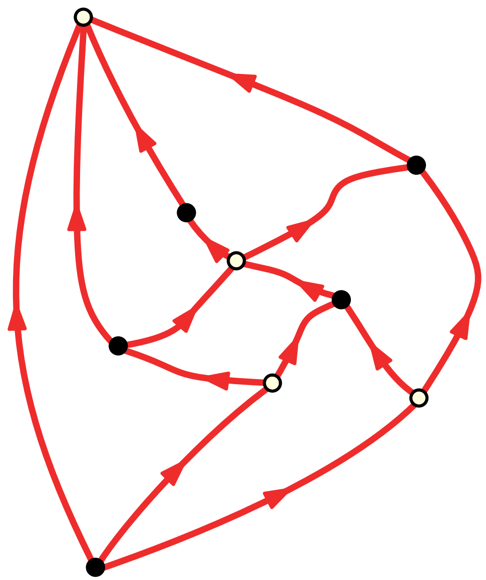
encoded by left-to-right bipolar orientation

Characterization of the bipolar orientation

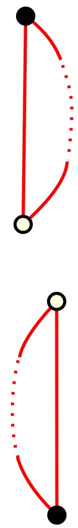
[F,Narmanli,Schaeffer'22]



$\simeq$



forbidden



Characterization of the bipolar orientation

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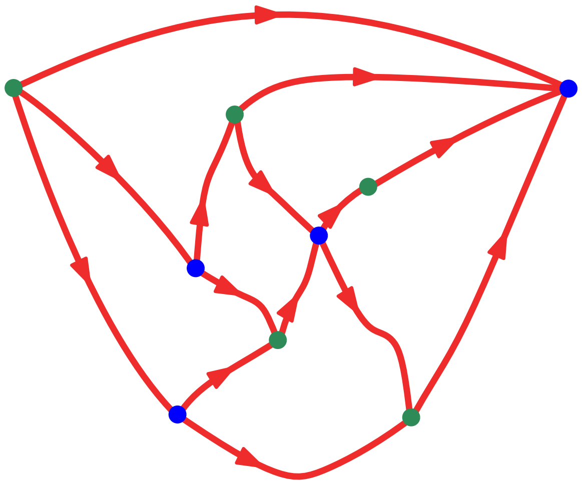
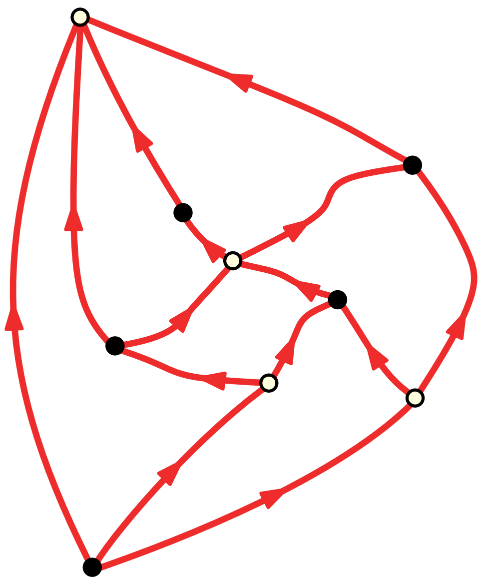
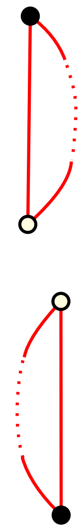


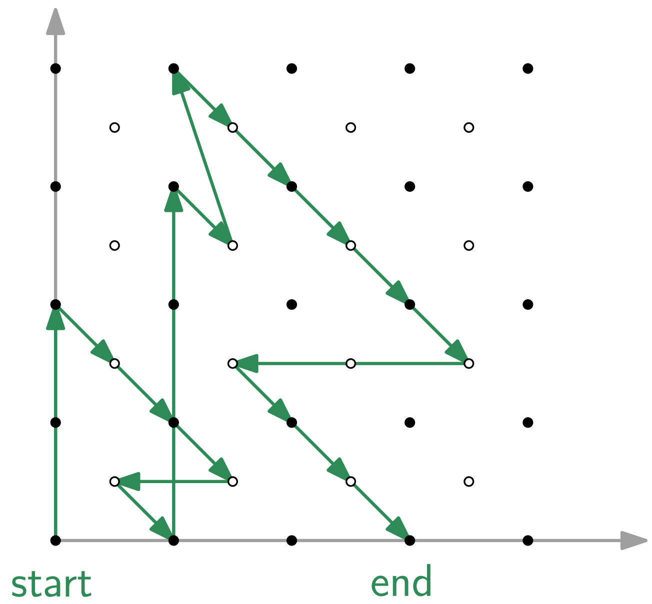
$\simeq$



forbidden



Corresponding quadrant tandem walks (bimodal effect)



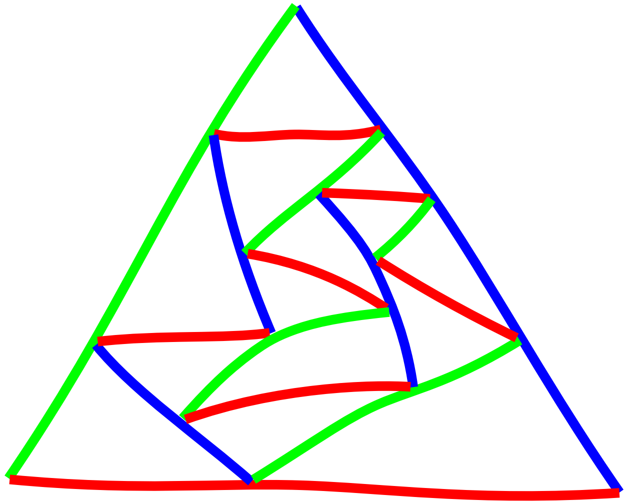
starts at 0, ends on x-axis

visits only points with $x + y$ even

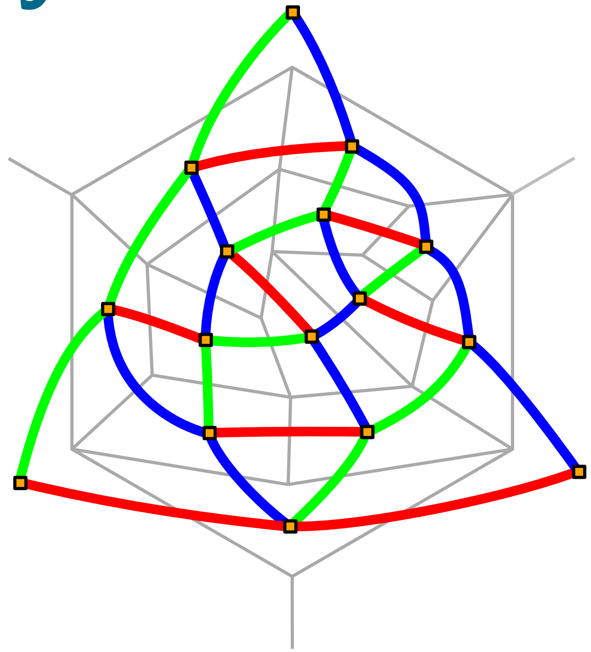
no horizontal step starting from ●

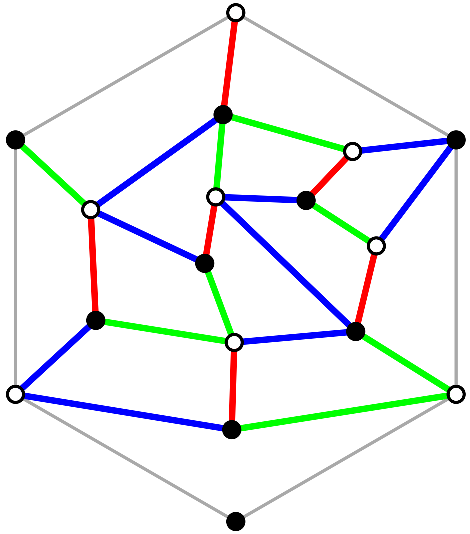
no vertical step starting from ○

Strong tricolored systems



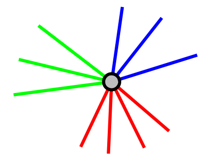
strong contact-system



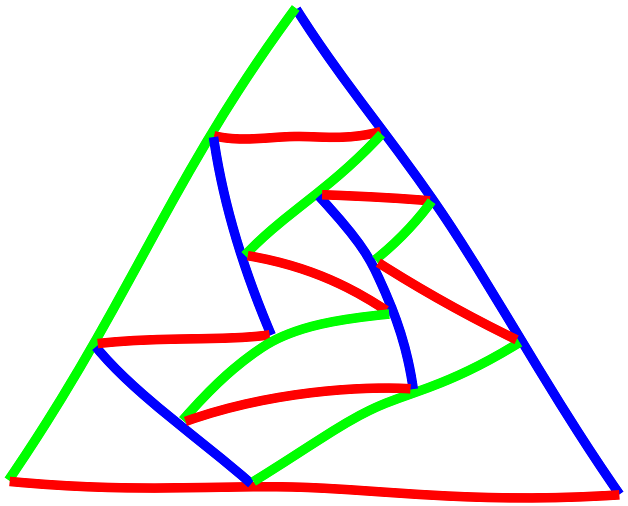


quadrangulation of hexagon
+ edge-tricoloration

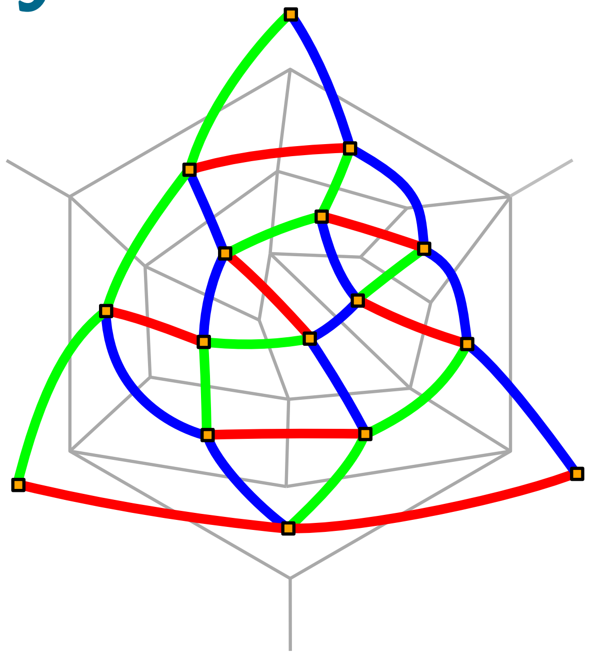
satisfying

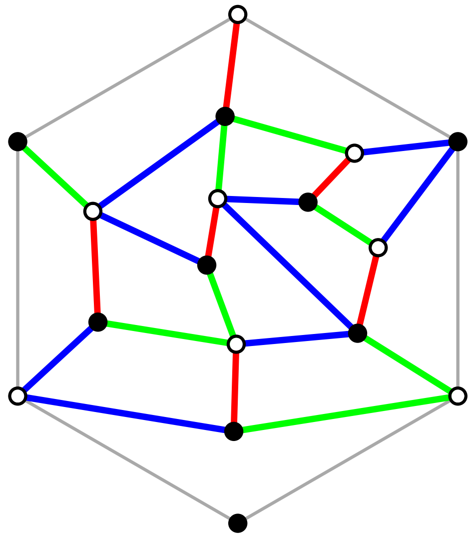


Strong tricolored systems



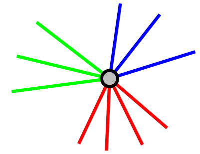
strong contact-system

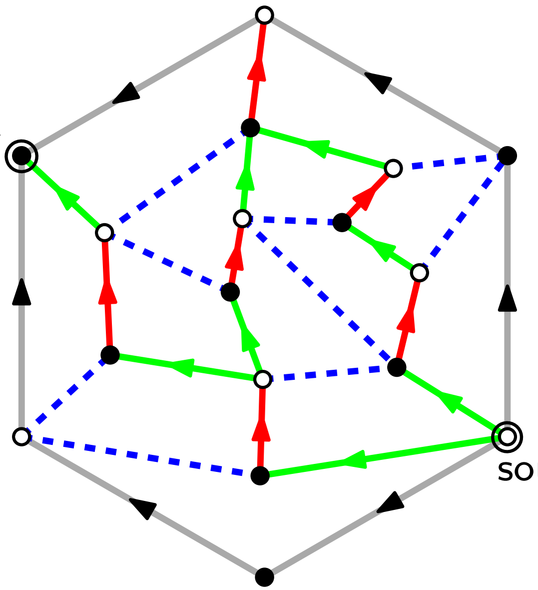


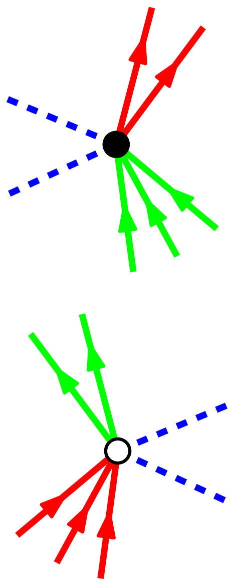


quadrangulation of hexagon
+ edge-tricoloration

satisfying







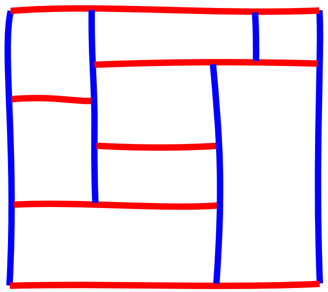
bipartite bipolar orientation
+ transversal edges

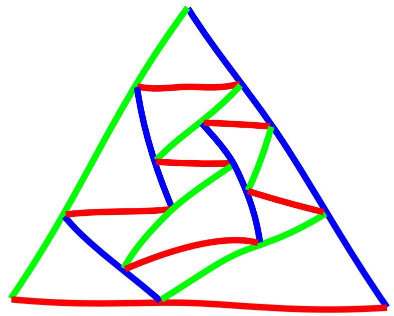
tandem walks have a bimodal condition + binomial weights

Asymptotic enumeration (updated)

Asymptotic estimate $c\,\gamma^n\,n^{-\alpha}$

$\alpha = 1 + \frac{\pi}{\theta}$

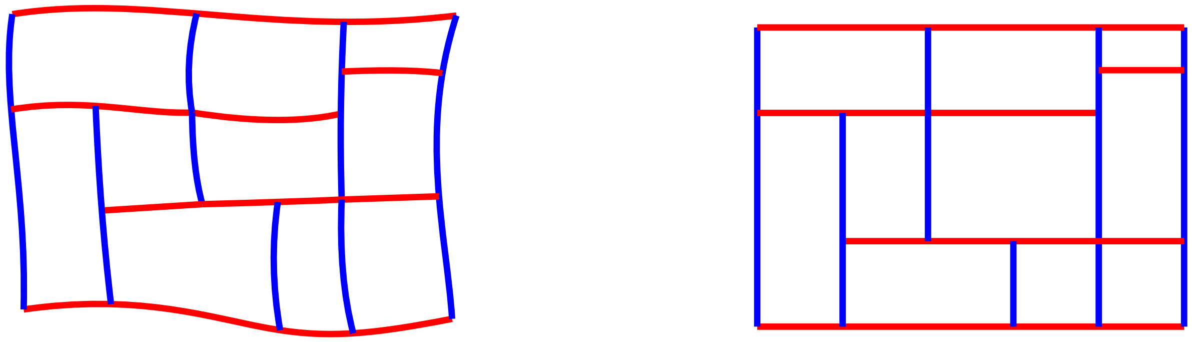


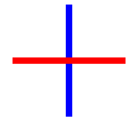


	weak	strong	weak	strong
	bipolar orientations	transversal structures	polyhedral orientations	(3c) Schnyder labelings
γ	8	$27/2$	$9/2$	$16/3$
$\cos(\theta)$	$1/2$	$7/8$	$9/16$ (*)	$22/27$ (*)
α	4	$\approx 7.21 \notin \mathbb{Q}$	$\approx 4.23 \notin \mathbb{Q}$	$\approx 6.08 \notin \mathbb{Q}$

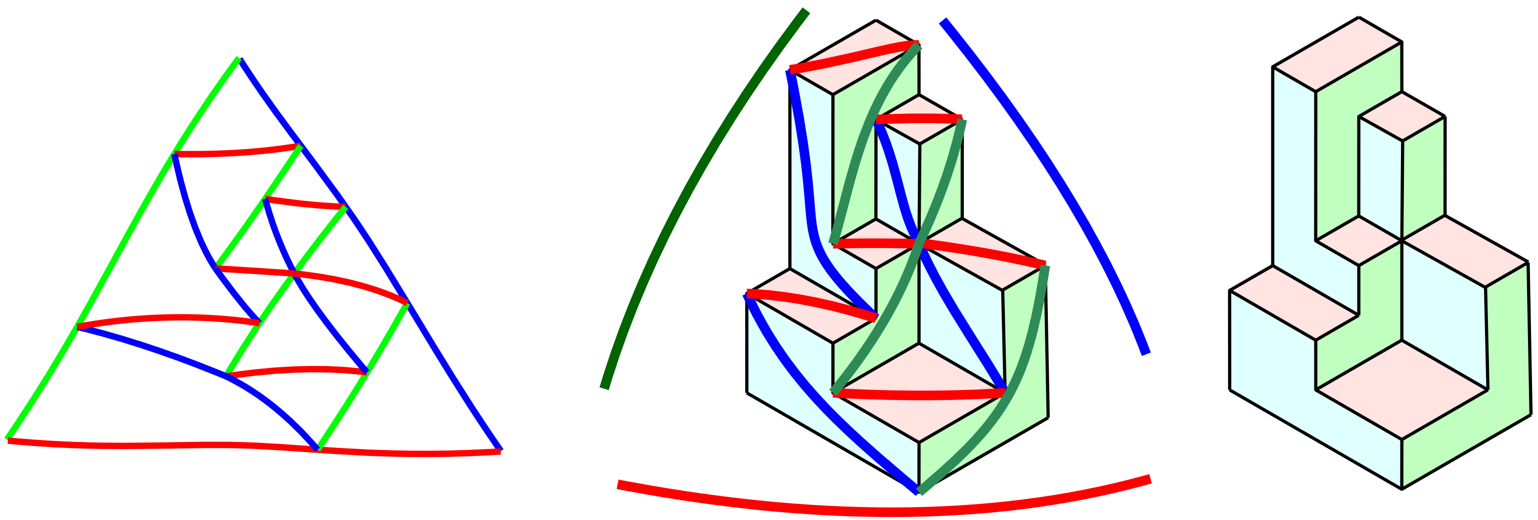
(*) up to extending [Denisov-Wachtel] to bimodal setting

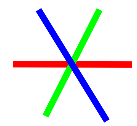
Extension to models with degeneracies



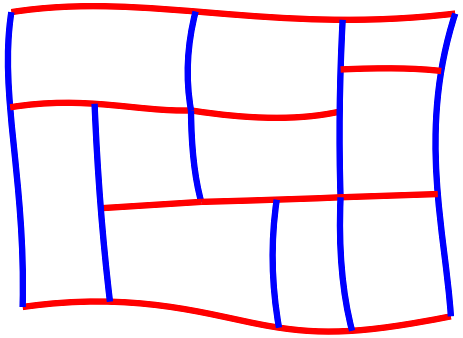
weight v per 

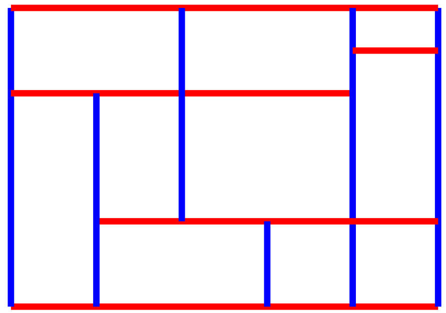
also counted in [Conant,Michaels'12]

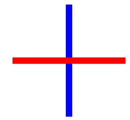


weight v per 

Extension to models with degeneracies





weight v per 

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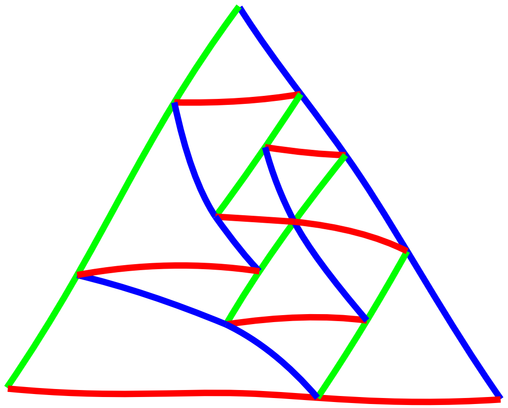
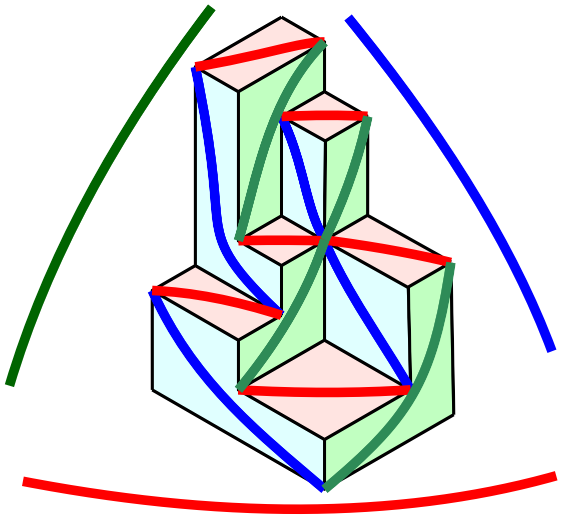
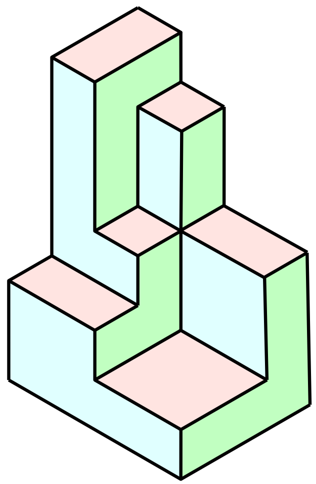
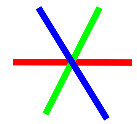
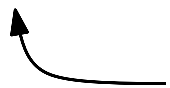






weight v per 

Asymptotic exponent $\alpha(v)$ computable $\alpha(v) \to \infty$ as $v \to \infty$



regular grid behaviour