Bijections between planar maps and planar linear normal λ-terms with connectivity condition

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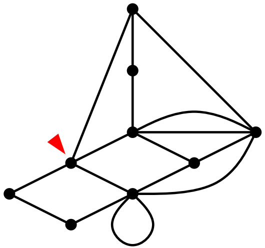
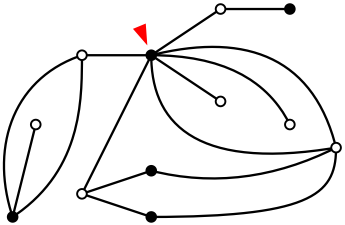
What is a planar map?

Planar map: drawings of graphs on a plane without extra crossing



planar



bipartite planar

Planar maps are rooted, *i.e.*, with a marked corner on outer face.

A crash course on λ-calculus

A **λ-term** is:

- Atom: variables $x, y, z, \ldots$;
- Application $t\ u$: apply t (as a function) to u;
- Abstraction $\lambda x.t$: build a function of x from t.

λ-calculus, which is Turing complete, consists of:

- α-renaming: variable names not important;
- **β-reduction**: $(\lambda x.t)\ u \to_\beta t[x \leftarrow u]$.

Example: $\Omega = (\lambda x.\ x\,x)(\lambda x.\ x\,x) \to_\beta (\lambda x.\ x\,x)(\lambda x.\ x\,x) = \Omega$.

β-reduction has a unique normal form if there is one.

Example: $(\lambda x.y)\ \Omega \to_\beta (\lambda x.y)\ \Omega \to_\beta \cdots \to_\beta (\lambda x.y)\ \Omega \to_\beta y$.

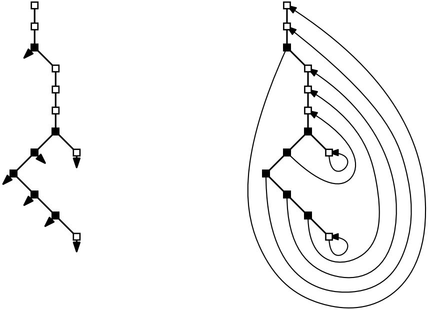
λ-terms, according to a combinatorialist

λ-term: unary-binary tree (skeleton) + variable-abstraction map

A variable is bound by an ancestral abstraction.

Linear λ-term: the variable-abstraction map being bijective

$$t = \lambda u.\lambda v.u(\lambda w.\lambda x.\lambda z.v(w(x(\lambda y.y)))z(\lambda k.k))$$



(RL-)planar term: counter-clockwise variable-abstraction map

Linear planar : unique choice, so just unary-binary tree!

Curry–Howard correspondence

Typed programs are **mathematical proofs**.

Computational side	**Logical side**
Typed λ-calculi	Logic systems
Types	Provable formulas
λ-terms	Proofs

- Typed λ-calculi avoids infinite computation (such as Ω).
- Linear λ-calculus $\Leftrightarrow$ "λ-fragment" of intuitionistic linear logic
- Variable as resource, abstraction as consumption

How exactly?

Known enumeration of various families of λ-terms

- unitless: each sub-term has at least one free variable
- normal: no possible β-reduction (so computed result)

λ-terms	Maps	OEIS
linear	general cubic	A062980
planar	planar cubic	A002005
unitless	bridgeless cubic	A267827
unitless planar	bridgeless planar cubic	A000309
β-normal linear/$\sim$	general	A000698
β-normal planar	planar	A000168
β-normal unitless linear/$\sim$	bridgeless	A000699
β-normal unitless planar	bridgeless planar	A000260

Noam Zeilberger, *A theory of linear typings as flows on 3-valent graphs*, LICS 2018

A lot of people and work: Bodini, Courtiel, Gardy, Giorgetti, Jacquot, Yeats, Zeilberger, . . .

Connectivity condition

Motivated (mysteriously) by type theory...

Noam Zeilberger and Jason Reed (Workshop CLA 2019)

How about connectivity of the diagram on **planar linear normal terms**?

k-connected: breaking $k-1$ edges does not split the graph

- 1-connected: all (connected by their skeleton)
- 2-connected: unitless (bridge ⇔ sub-term without free variable)
- 3-connected: ??? (characterized by type theory for general k)

Conjecture (Zeilberger–Reed, 2019)

The number of 3-connected planar linear normal λ-terms with $n+2$ variables is

$$\frac{2^n}{(n+1)(n+2)}\binom{2n+1}{n},$$

which also counts bipartite planar maps with n edges (A000257).

Our contribution (1)

Katarzyna Grygiel and Guan-Ru Yu (Workshop CLA 2020): combinatorial characterization of 3-connected terms, partial bijective results

Theorem (F. 2023)

There is a direct bijection between 3-connected planar linear normal λ-terms with $n+2$ variables and bipartite planar maps with n edges.

What we do using bijections:

- Transfer of statistics
- Generating functions and probabilistic results also for free!

Proposition (F. 2023, from known results on maps by Liskovet)

Let $X_n = \#$ initial abstractions of a uniformly random 3-connected planar linear normal λ-term. When $n \to \infty$,

$$\mathbb{P}[X_n = k] \to \frac{k-1}{3} \binom{2k-2}{k-1} \left(\frac{3}{16}\right)^{k-1}.$$

Our contribution (2)

Theorem (F. 2023)

There is a direct bijection from planar linear normal λ-terms to planar maps, with its restriction to unitless terms giving loopless planar maps.

λ-terms	Maps	OEIS
β-normal linear/$\sim$	general	A000698
β-normal planar	planar	A000168
β-normal unitless linear/$\sim$	bridgeless	A000699
β-normal unitless planar	loopless planar	A000260

Known recursive bijection in (Zeilberger and Giorgetti, 2015) via LR-planar terms (clockwise, not stable by β-reduction...)

From λ-terms to unary-binary trees

Linear planar λ-terms $\Leftrightarrow$ unary-binary trees (with conditions)

Three statistics for a unary-binary tree S:

- $\mathrm{unary}(S)$: # unary nodes (abstractions)
- $\mathrm{leaf}(S)$: # leaves (variables)
- $\mathrm{excess}(S)$: $\mathrm{leaf}(S) - \mathrm{unary}(S)$ (free variables, *i.e.*, not yet bound)

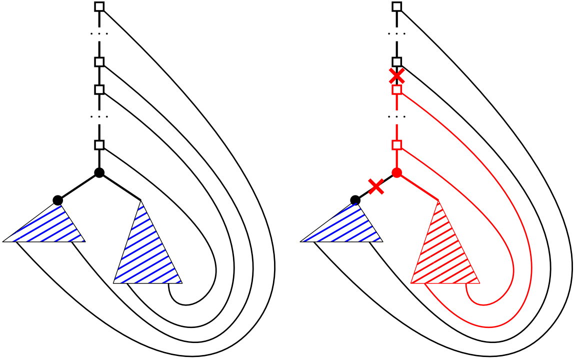
S_u: sub-tree of S induced by u

- Linear $\Leftrightarrow$ $\mathrm{excess}(S) = 0$
- 1-connected (well-scoped) $\Leftrightarrow$ $\mathrm{excess}(S_u) \geq 0$ for all u
- 2-connected (or unitless) $\Leftrightarrow$ $\mathrm{excess}(S_u) > 0$ for all u non-root

Characterization of 3-connectedness (1)

Proposition (Grygiel and Yu, CLA 2020)

In the skeleton of a 3-connected planar linear λ-term, the left child of the first binary node is a leaf.



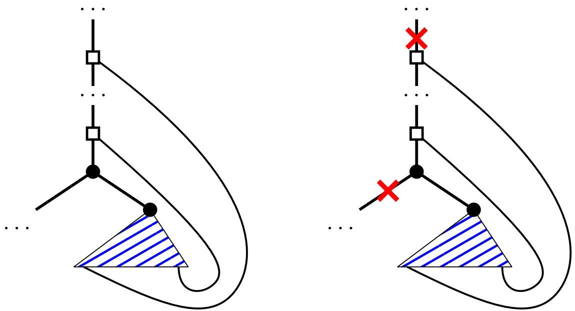
Reduced skeleton: the right sub-tree of the first binary node

Characterization of 3-connectedness (2)

Proposition (Proposed by Grygiel and Yu, CLA 2020)

S is the reduced skeleton of a 3-connected planar linear normal λ-term iff

- **(Normality)** *The left child of a binary node in S is never unary;*
- **(3-connectedness)** *For every binary node u with v its right child, # consecutive unary nodes above $u < \mathrm{excess}(S_v)$.*



Clearly necessary, but also sufficient!

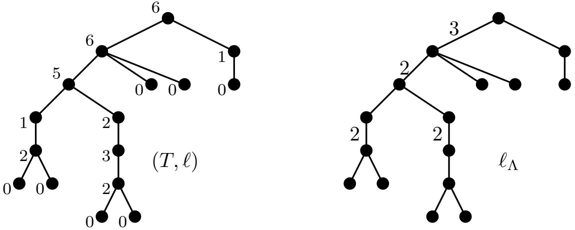
Degree trees

Degree tree: a plane tree T with a labeling ℓ on nodes with

- u is a leaf $\Rightarrow \ell(u) = 0$;
- u has children $v_1, \dots v_k \Rightarrow s(u) - \ell(v_1) \leq \ell(u) \leq s(u)$, where $s(u) = k + \sum_{i=1}^{k} \ell(v_i)$.

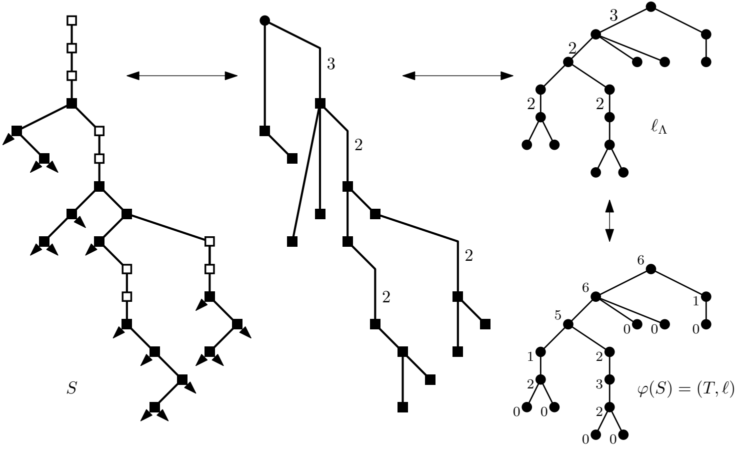
Contribution of each child : 1 (itself) $+ \ell(v_i)$ (its label)

Except for the first child: from 1 to its due contribution.



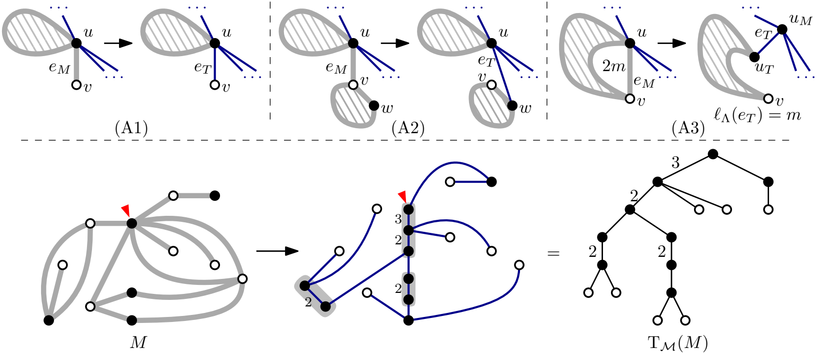
Edge labeling ℓ_Λ: the retained contribution (interchangeable with ℓ !)

Bijection (1/2): 3-connected terms $\Leftrightarrow$ degree trees



- Contribution of u to parent $(1 + \ell(u))$ = excess of its right sub-tree
- Unary nodes on right child $\Leftrightarrow$ retainment by left child

Bijection (2/2): degree trees ⇔ bipartite planar maps



Existing direct bijection (F., 2021), as an exploration process

Also in bijection with Chapoton's new intervals in the Tamari lattice

Some statistics correspondences:

- Unary chains of length k ⇔ edge label k ⇔ inner faces of degree $2k$
- Initial unary chain ⇔ root label ⇔ degree of root face

Conclusion

The second bijection has a similar flavor.

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β-normal 3-connected planar	bipartite planar	A000257

Higher connectivity? Other enumeration consequences?

And (principal) types (1, 2, 9, 52, 344, 2482, 19028, 152570, ...) ?

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Higher connectivity? Other enumeration consequences?

And (principal) types (1, 2, 9, 52, 344, 2482, 19028, 152570, ...) ?

Thank you for listening!

Curry–Howard correspondence

Simply typed λ-calculus

Context/Premise $\vdash$ inferred type of term

$$\frac{}{x : \alpha \vdash x : \alpha}(\mathsf{Ax}) \quad \frac{\Gamma, x : \alpha \vdash t : \beta}{\Gamma \vdash \lambda x.t : \alpha \to \beta}(\to_I) \quad \frac{\Gamma \vdash t : \alpha \to \beta \quad \Delta \vdash u : \alpha}{\Gamma, \Delta \vdash t\ u : \beta}(\to_E)$$

$$\frac{\Gamma, \Delta \vdash u : \alpha}{\Gamma, t : \beta, \Delta \vdash u : \alpha}(\text{weakening}) \quad \frac{\Gamma, t : \beta, t : \beta, \Delta \vdash u : \alpha}{\Gamma, t : \beta, \Delta \vdash u : \alpha}(\text{contraction})$$

$$\frac{\Gamma, t : \beta, s : \gamma, \Delta \vdash u : \alpha}{\Gamma, s : \gamma, t : \beta, \Delta \vdash u : \alpha}(\text{exchange})$$

Curry–Howard correspondence

Simply typed λ-calculus $\Leftrightarrow$ Intuitionistic implicational natural deduction

Context/Premise $\vdash$ deduced formula

$$\frac{}{x : \alpha \vdash x : \alpha}(\text{Ax}) \quad \frac{\Gamma, x : \alpha \vdash t : \beta}{\Gamma \vdash \lambda x.t : \alpha \to \beta}(\to_I) \quad \frac{\Gamma \vdash t : \alpha \to \beta \quad \Delta \vdash u : \alpha}{\Gamma, \Delta \vdash t\ u : \beta}(\to_E)$$

$$\frac{\Gamma, \Delta \vdash u : \alpha}{\Gamma, t : \beta, \Delta \vdash u : \alpha}(\text{weakening}) \quad \frac{\Gamma, t : \beta, t : \beta, \Delta \vdash u : \alpha}{\Gamma, t : \beta, \Delta \vdash u : \alpha}(\text{contraction})$$

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Curry–Howard correspondence

Constrained λ-calculus $\Leftrightarrow$ Substructural intuitionistic logic

Context/Premise $\vdash$ inferred type of term deduced formula

$$\frac{}{x : \alpha \vdash x : \alpha}(\text{Ax}) \quad \frac{\Gamma, x : \alpha \vdash t : \beta}{\Gamma \vdash \lambda x.t : \alpha \to \beta}(\to_I) \quad \frac{\Gamma \vdash t : \alpha \to \beta \quad \Delta \vdash u : \alpha}{\Gamma, \Delta \vdash t\ u : \beta}(\to_E)$$

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$$\frac{\Gamma, t : \beta, s : \gamma, \Delta \vdash u : \alpha}{\Gamma, s : \gamma, t : \beta, \Delta \vdash u : \alpha}(\text{exchange})$$

~~(weakening)~~, ~~(contraction)~~ $\Rightarrow$ linear, ~~(exchange)~~ $\Rightarrow$ planar

"λ-fragment" of linear logic, related to some programming languages

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