Bijections between planar maps and planar linear normal λ-terms with connectivity condition

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What is a planar map?

Planar map: drawings of graphs on a plane without extra crossing

Planar maps are rooted, i.e., with a marked corner on outer face.
A crash course on λ-calculus

A λ-term is:

- **Atom**: variables $x, y, z, \ldots$;
- **Application** $t u$: apply $t$ (as a function) to $u$;
- **Abstraction** $\lambda x.t$: build a function of $x$ from $t$.

**λ-calculus**, which is Turing complete, consists of:

- **α-renaming**: variable names not important;
- **β-reduction**: $(\lambda x.t) u \rightarrow_{\beta} t[x \leftarrow u]$.

Example: $\Omega = (\lambda x. x x)(\lambda x. x x) \rightarrow_{\beta} (\lambda x. x x)(\lambda x. x x) = \Omega$.

β-reduction has a **unique normal form** if there is one.

Example: $(\lambda x.y) \Omega \rightarrow_{\beta} (\lambda x.y) \Omega \rightarrow_{\beta} \cdots \rightarrow_{\beta} (\lambda x.y) \Omega \rightarrow_{\beta} y$. 
\( \lambda \)-terms, according to a combinatorialist

\textbf{\( \lambda \)-term}: unary-binary tree (skeleton) + variable-abstraction map

A variable is \textbf{bound} by an ancestral abstraction.

\textbf{Linear \( \lambda \)-term}: the variable-abstraction map being \textbf{bijective}

\[
t = \lambda u. \lambda v. u (\lambda w. \lambda x. \lambda z. v (w (x (\lambda y. y)))) z (\lambda k. k)
\]

\textbf{(RL-)planar term}: counter-clockwise variable-abstraction map

Linear planar: unique choice, so just unary-binary tree!
Curry–Howard correspondence

Typed programs are mathematical proofs.

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- Typed $\lambda$-calculi avoids infinite computation (such as $\Omega$).
- Linear $\lambda$-calculus $\Leftrightarrow$ “$\lambda$-fragment” of intuitionistic linear logic
- Variable as resource, abstraction as consumption

How exactly?
Known enumeration of various families of $\lambda$-terms

- **unitless**: each sub-term has at least one free variable
- **normal**: no possible $\beta$-reduction (so computed result)

<table>
<thead>
<tr>
<th>$\lambda$-terms</th>
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<th>OEIS</th>
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<tbody>
<tr>
<td>linear</td>
<td>general cubic</td>
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</tr>
<tr>
<td>planar</td>
<td>planar cubic</td>
<td>A002005</td>
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<tr>
<td>unitless</td>
<td>bridgeless cubic</td>
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<td>unitless planar</td>
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<td>$\beta$-normal linear/∼</td>
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Noam Zeilberger, *A theory of linear typings as flows on 3-valent graphs*, LICS 2018

A lot of people and work: Bodini, Courtiel, Gardy, Giorgetti, Jacquot, Yeats, Zeilberger, ...
Connectivity condition

Motivated (mysteriously) by type theory...

Noam Zeilberger and Jason Reed (Workshop CLA 2019)

How about connectivity of the diagram on planar linear normal terms?

$k$-connected: breaking $k - 1$ edges does not split the graph

- 1-connected: all (connected by their skeleton)
- 2-connected: unitless (bridge $\Leftrightarrow$ sub-term without free variable)
- 3-connected: ??? (characterized by type theory for general $k$)

Conjecture (Zeilberger–Reed, 2019)

The number of 3-connected planar linear normal $\lambda$-terms with $n + 2$ variables is

$$\frac{2^n}{(n + 1)(n + 2)} \binom{2n + 1}{n},$$

which also counts bipartite planar maps with $n$ edges (A000257).
Our contribution (1)

Katarzyna Grygiel and Guan-Ru Yu (Workshop CLA 2020): combinatorial characterization of 3-connected terms, partial bijective results

Theorem (F. 2023)

There is a direct bijection between 3-connected planar linear normal λ-terms with \( n + 2 \) variables and bipartite planar maps with \( n \) edges.

What we do using bijections:

- Transfer of statistics
- Generating functions and probabilistic results also for free!

Proposition (F. 2023, from known results on maps by Liskovet)

Let \( X_n = \# \) initial abstractions of a uniformly random 3-connected planar linear normal λ-term. When \( n \to \infty \),

\[
\mathbb{P}[X_n = k] \to \frac{k - 1}{3} \binom{2k - 2}{k - 1} \left( \frac{3}{16} \right)^{k-1}.
\]
Our contribution (2)

Theorem (F. 2023)

There is a direct bijection from planar linear normal $\lambda$-terms to planar maps, with its restriction to unitless terms giving loopless planar maps.

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Known recursive bijection in (Zeilberger and Giorgetti, 2015) via LR-planar terms (clockwise, not stable by $\beta$-reduction...
From $\lambda$-terms to unary-binary trees

Linear planar $\lambda$-terms $\Leftrightarrow$ unary-binary trees (with conditions)

Three statistics for a unary-binary tree $S'$:
- $\text{unary}(S')$: $\#$ unary nodes (abstractions)
- $\text{leaf}(S')$: $\#$ leaves (variables)
- $\text{excess}(S')$: $\text{leaf}(S') - \text{unary}(S')$ (free variables, i.e., not yet bound)

$S_u$: sub-tree of $S$ induced by $u$
- Linear $\Leftrightarrow$ $\text{excess}(S) = 0$
- 1-connected (well-scoped) $\Leftrightarrow$ $\text{excess}(S_u) \geq 0$ for all $u$
- 2-connected (or unitless) $\Leftrightarrow$ $\text{excess}(S_u) > 0$ for all $u$ non-root
Characterization of 3-connectedness (1)

Proposition (Grygiel and Yu, CLA 2020)

*In the skeleton of a 3-connected planar linear \( \lambda \)-term, the left child of the first binary node is a leaf.*

Reduced skeleton: the right sub-tree of the first binary node
Characterization of 3-connectedness (2)

Proposition (Proposed by Grygiel and Yu, CLA 2020)

\( S \) is the **reduced skeleton** of a 3-connected planar linear normal \( \lambda \)-term iff

- **(Normality)** The left child of a binary node in \( S \) is never unary;
- **(3-connectedness)** For every binary node \( u \) with \( v \) its right child, \( \# \) consecutive unary nodes above \( u \) < \( \text{excess}(S_v) \).

Clearly necessary, but also sufficient!
Degree trees

**Degree tree**: a plane tree $T$ with a labeling $\ell$ on nodes with
- $u$ is a leaf $\Rightarrow \ell(u) = 0$;
- $u$ has children $v_1, \ldots, v_k \Rightarrow s(u) - \ell(v_1) \leq \ell(u) \leq s(u)$, where $s(u) = k + \sum_{i=1}^{k} \ell(v_i)$.

**Contribution of each child**: $1$ (itself) + $\ell(v_i)$ (its label)

**Except for the first child**: from $1$ to its due contribution.

Edge labeling $\ell_\Lambda$: the retained contribution (interchangeable with $\ell$ !)
Bijection (1/2): 3-connected terms $\Leftrightarrow$ degree trees

- Contribution of $u$ to parent $(1 + \ell(u)) = \text{excess of its right sub-tree}$
- Unary nodes on right child $\Leftrightarrow$ retainment by left child
Bijection (2/2): degree trees $\Leftrightarrow$ bipartite planar maps

Existing direct bijection (F., 2021), as an exploration process

Also in bijection with Chapoton’s new intervals in the Tamari lattice

Some statistics correspondences:

- Unary chains of length $k$ $\Leftrightarrow$ edge label $k$ $\Leftrightarrow$ inner faces of degree $2k$
- Initial unary chain $\Leftrightarrow$ root label $\Leftrightarrow$ degree of root face
The second bijection has a similar flavor.

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Higher connectivity? Other enumeration consequences?

And (principal) types (1, 2, 9, 52, 344, 2482, 19028, 152570, ...)?
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Higher connectivity? Other enumeration consequences?

And (principal) types (1, 2, 9, 52, 344, 2482, 19028, 152570, ...) ?

Thank you for listening!
Curry–Howard correspondence

Simply typed λ-calculus

Context/Premise ⊢ inferred type of term

\[
\begin{align*}
(Ax) & \quad x : \alpha \vdash x : \alpha \\
\Gamma, x : \alpha \vdash t : \beta & \quad \Gamma \vdash \lambda x.t : \alpha \to \beta \quad (\to_I) \\
\Gamma \vdash t : \alpha \to \beta & \quad \Delta \vdash u : \alpha \quad \Gamma, \Delta \vdash t u : \beta \quad (\to_E) \\
\Gamma, \Delta \vdash u : \alpha & \quad \text{(weakening)} \\
\Gamma, t : \beta, \Delta \vdash u : \alpha & \quad \text{(contraction)} \\
\Gamma, t : \beta, s : \gamma, \Delta \vdash u : \alpha & \quad \text{(exchange)}
\end{align*}
\]
Curry–Howard correspondence

Simply typed $\lambda$-calculus $\iff$ Intuitionistic implicational natural deduction

Context/Premise $\vdash$ deduced formula

\[
\begin{align*}
\frac{}{x : \alpha \vdash x : \alpha} \quad (Ax) \\
\frac{\Gamma, x : \alpha \vdash t : \beta}{\Gamma \vdash \lambda x.t : \alpha \rightarrow \beta} \quad (\rightarrow_I) \\
\frac{\Gamma \vdash t : \alpha \rightarrow \beta}{\Gamma, \Delta \vdash t \, u : \beta} \quad (\rightarrow_E)
\end{align*}
\]

\[
\begin{align*}
\frac{\Gamma, \Delta \vdash u : \alpha}{\Gamma, t : \beta, \Delta \vdash u : \alpha} \quad \text{(weakening)} \\
\frac{\Gamma, t : \beta, \Delta \vdash u : \alpha}{\Gamma, t : \beta, \Delta \vdash u : \alpha} \quad \text{(contraction)} \\
\frac{\Gamma, t : \beta, s : \gamma, \Delta \vdash u : \alpha}{\Gamma, s : \gamma, t : \beta, \Delta \vdash u : \alpha} \quad \text{(exchange)}
\end{align*}
\]
Curry–Howard correspondence

Constrained λ-calculus ⇔ Substructural intuitionistic logic

Context/Premise ⊢ inferred type of term deduced formula

\[
\begin{align*}
\text{Ax} & : x : \alpha \vdash x : \alpha \\
\frac{}{\Gamma \vdash \lambda x. t : \alpha \rightarrow \beta} & (\rightarrow I) \\
\frac{}{\Gamma \vdash t : \alpha \rightarrow \beta \quad \Delta \vdash u : \alpha} & (\rightarrow E)
\end{align*}
\]

\[
\begin{align*}
\text{weakening} & : \Gamma, \Delta \vdash u : \alpha \quad \text{weakening} \quad \Gamma, t : \beta, \Delta \vdash u : \alpha \\
\text{contraction} & : \Gamma, t : \beta, t : \beta, \Delta \vdash u : \alpha \quad \text{contraction} \quad \Gamma, t : \beta, \Delta \vdash u : \alpha \\
\text{exchange} & : \Gamma, t : \beta, s : \gamma, \Delta \vdash u : \alpha \quad \text{exchange} \quad \Gamma, s : \gamma, t : \beta, \Delta \vdash u : \alpha
\end{align*}
\]

(weakening), (contraction) ⇒ linear, (exchange) ⇒ planar

“λ-fragment” of linear logic, related to some programming languages