# On the $A_2$ Andrews–Schilling–Warnaar Identities

Shashank Kanade

University of Denver

Based on joint work with Matthew C. Russell

# Cast and the Characters

# Rogers–Ramanujan identities

# Rogers–Ramanujan identities

$$\sum_{n\geq 0} \frac{q^{n^2}}{(q)_n} = \frac{1}{(q,q^4;q^5)_\infty}$$

$$\sum_{n\geq 0} \frac{q^{n^2+n}}{(q)_n} = \frac{1}{(q^2,q^3;q^5)_\infty}$$

# Rogers–Ramanujan identities

$$\sum_{n\geq 0} \frac{q^{n^2}}{(q)_n} = \frac{1}{(q,q^4;q^5)_\infty}$$

$$\sum_{n\geq 0} \frac{q^{n^2+n}}{(q)_n} = \frac{1}{(q^2,q^3;q^5)_\infty}$$

VOAs, affine Lie algebras, and related structures

## Rogers–Ramanujan identities

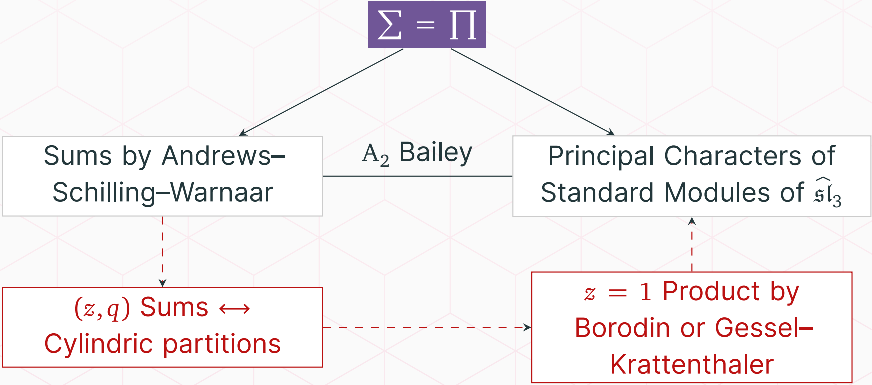
$$\sum_{n\geq 0} \frac{q^{n^2}}{(q)_n} = \frac{1}{(q,q^4;q^5)_\infty}$$

$$\sum_{n\geq 0} \frac{q^{n^2+n}}{(q)_n} = \frac{1}{(q^2,q^3;q^5)_\infty}$$

VOAs, affine Lie algebras, and related structures

Knot Theory

# $A_2$ Andrews–Schilling–Warnaar identities



# Setup

# Principal Characters: $\widehat{\mathfrak{sl}}_3$

# Principal Characters: $\widehat{\mathfrak{sl}}_3$

$\lambda = c_0\Lambda_0 + c_1\Lambda_1 + c_2\Lambda_2 \ (c_i \in \mathbb{Z}_{\geq 0})$

## Principal Characters: $\widehat{\mathfrak{sl}}_3$

$\lambda = c_0\Lambda_0 + c_1\Lambda_1 + c_2\Lambda_2 \ (c_i \in \mathbb{Z}_{\geq 0})$

Level = $\ell = c_0 + c_1 + c_2$

# Principal Characters: $\widehat{\mathfrak{sl}}_3$

$\lambda = c_0\Lambda_0 + c_1\Lambda_1 + c_2\Lambda_2 \ (c_i \in \mathbb{Z}_{\geq 0})$

Level = $\ell = c_0 + c_1 + c_2$

Modulus = $m = \ell + 3 = c_0 + c_1 + c_2 + 3$

# Principal Characters: $\widehat{\mathfrak{sl}}_3$

$\lambda = c_0\Lambda_0 + c_1\Lambda_1 + c_2\Lambda_2 \ (c_i \in \mathbb{Z}_{\geq 0})$

Level = $\ell = c_0 + c_1 + c_2$

Modulus = $m = \ell + 3 = c_0 + c_1 + c_2 + 3$

Principally specialized character:

$\chi(L(\lambda)) = \left(e^{-\lambda}\,\mathrm{ch}(L(\lambda))\right)|_{e^{-\alpha_i}\mapsto q}$

# Principal Characters: $\widehat{\mathfrak{sl}}_3$

$\lambda = c_0\Lambda_0 + c_1\Lambda_1 + c_2\Lambda_2 \ (c_i \in \mathbb{Z}_{\geq 0})$

Level = $\ell = c_0 + c_1 + c_2$

Modulus = $m = \ell + 3 = c_0 + c_1 + c_2 + 3$

Principally specialized character:

$\chi(L(\lambda)) = \left(e^{-\lambda}\operatorname{ch}(L(\lambda))\right)|_{e^{-\alpha_i}\mapsto q}$

## Principal character

$$\chi(\Omega(\lambda)) = \frac{\chi(L(\lambda))}{\chi(L(\Lambda_0))}$$

# Principal Characters: $\widehat{\mathfrak{sl}}_3$

$\lambda = c_0\Lambda_0 + c_1\Lambda_1 + c_2\Lambda_2 \ (c_i \in \mathbb{Z}_{\geq 0})$

Level = $\ell = c_0 + c_1 + c_2$

Modulus = $m = \ell + 3 = c_0 + c_1 + c_2 + 3$

Principally specialized character:

$\chi(L(\lambda)) = \left(e^{-\lambda}\,\mathrm{ch}(L(\lambda))\right)|_{e^{-\alpha_i}\mapsto q}$

## Principal character

$$\chi(\Omega(\lambda)) = \frac{\chi(L(\lambda))}{\chi(L(\Lambda_0))}$$

$$= \frac{(q^m;q^m)_\infty^2}{(q)_\infty^2}(q^{1+c_0}, q^{m-1-c_0}, q^{1+c_1}, q^{m-1-c_1}, q^{1+c_2}, q^{m-1-c_2}; q^m)_\infty$$

# Arrangement of Modules

# Arrangement of Modules

Level 11

## Arrangement of Modules

Level 11

| | | | |
|---|---|---|---|
| 11, 0, 0 | | | |
| 10, 1, 0 | 9, 1, 1 | | |
| 9, 2, 0 | 8, 2, 1 | 7, 2, 2 | |
| 8, 3, 0 | 7, 3, 1 | 6, 3, 2 | 5, 3, 3 |
| 7, 4, 0 | 6, 4, 1 | 5, 4, 2 | 4, 4, 3 |
| 6, 5, 0 | 5, 5, 1 | | |

# Sample identities (ASW)

## Sample identities (ASW)

### Level 7 (Andrews–Schilling–Warnaar)

$$\sum_{\substack{r_1\geq r_2\geq 0\\ s_1\geq s_2\geq 0}} \frac{q^{(r_1^2-r_1s_1+s_1^2)+(r_2^2-r_2s_2+s_2^2)+r_1+r_2+s_1+s_2}}{(q)_{r_1-r_2}(q)_{s_1-s_2}(q)_{r_2}(q)_{s_2}(q)_{r_2+s_2+1}}$$

$$= \frac{1}{(q)_\infty}\chi(\Omega(7\Lambda_0+0\Lambda_1+0\Lambda_2))$$

$$= \frac{(q^{10};q^{10})_\infty^2}{(q)_\infty^3}(q,q,q^2,q^8,q^9,q^9;q^{10})_\infty$$

# Sample identities (ASW)

## Level 5 (Andrews–Schilling–Warnaar)

$$\sum_{\substack{r_1 \geq r_2 \geq 0 \\ s_1 \geq s_2 \geq 0}} \frac{q^{(r_1^2 - r_1 s_1 + s_1^2) + (r_2^2 + r_2 s_2 + s_2^2) + r_1 + r_2 + s_1 + s_2}}{(q)_{r_1 - r_2} (q)_{s_1 - s_2} (q)_{r_2} (q)_{s_2} (q)_{r_2 + s_2 + 1}}$$

$$= \frac{1}{(q)_\infty} \chi(\Omega(5\Lambda_0 + 0\Lambda_1 + 0\Lambda_2))$$

$$= \frac{(q^8; q^8)_\infty^2}{(q)_\infty^3} (q, q, q^2, q^6, q^7, q^7; q^8)_\infty$$

# Sample identities (ASW)

## Level 6 (Andrews–Schilling–Warnaar)

$$\sum_{\substack{r_1 \geq r_2 \geq 0 \\ s_1 \geq s_2 \geq 0}} \frac{q^{(r_1^2 - r_1 s_1 + s_1^2) + (r_2^2 - r_2 s_2 + s_2^2) + r_1 + r_2 + s_1 + s_2}}{(q)_{r_1 - r_2} (q)_{s_1 - s_2} (q)_{r_2 + s_2} (q)_{r_2 + s_2 + 1}} \begin{bmatrix} r_2 + s_2 \\ r_2 \end{bmatrix}_{q^3}$$

$$= \frac{1}{(q)_\infty} \chi(\Omega(6\Lambda_0 + 0\Lambda_1 + 0\Lambda_2))$$

$$= \frac{(q^9; q^9)_\infty^2}{(q)_\infty^3} (q, q, q^2, q^7, q^8, q^8; q^9)_\infty$$

## The main question

## The main question

### Question

Find and prove ASW identities for the remaining principal characters of $\widehat{\mathfrak{sl}}_3$.

# Cylindric Partitions

# Plane Partitions

## Plane Partitions

A plane partition

# Plane Partitions

A plane partition

| | | | | |
|---|---|---|---|---|
| 7 | 4 | 3 | 2 | 2 |
| 5 | 4 | 1 | | |
| 2 | | | | |

# Plane Partitions

A plane partition

| | | | | |
|---|---|---|---|---|
| 7 | 4 | 3 | 2 | 2 |
| 5 | 4 | 1 | | |
| 2 | | | | |

Numbers in boxes as heights

# Plane Partitions

A plane partition

| | | | | |
|---|---|---|---|---|
| 7 | 4 | 3 | 2 | 2 |
| 5 | 4 | 1 | | |
| 2 | | | | |

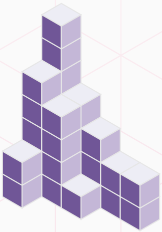
Numbers in boxes as heights → 3-dimensional shape

# Plane Partitions

A plane partition

| | | | | |
|---|---|---|---|---|
| 7 | 4 | 3 | 2 | 2 |
| 5 | 4 | 1 | | |
| 2 | | | | |

Numbers in boxes as heights → 3-dimensional shape



# Cylindric Partitions

# Cylindric Partitions

Cylindric partition of profile $c = (3, 1, 2)$

# Cylindric Partitions

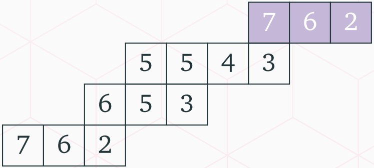
Cylindric partition of profile $c = (3, 1, 2)$

Connected to $\chi(\Omega(3\Lambda_0 + 1\Lambda_1 + 2\Lambda_2))$ for $\widehat{\mathfrak{sl}}_3 = A_2^{(1)}$

# Cylindric Partitions

Cylindric partition of profile $c = (3, 1, 2)$

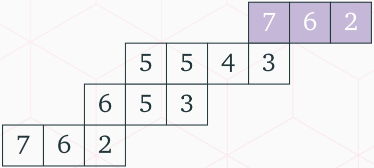
Connected to $\chi(\Omega(3\Lambda_0 + 1\Lambda_1 + 2\Lambda_2))$ for $\widehat{\mathfrak{sl}}_3 = A_2^{(1)}$



# Cylindric Partitions

Cylindric partition of profile $c = (3, 1, 2)$

Connected to $\chi(\Omega(3\Lambda_0 + 1\Lambda_1 + 2\Lambda_2))$ for $\widehat{\mathfrak{sl}}_3 = A_2^{(1)}$




Total “skew-ness” = level = $3 + 1 + 2 = 6$

Weight = $5 + 5 + 4 + 3 \; + \; 6 + 5 + 3 \; + \; 7 + 6 + 2$

Max-part = $7$

# Generating Functions

## Generating Functions

$c$: Profile

## Generating Functions

$c$: Profile

$\mathscr{C}_c$: Set of cylindric partitions of profile $c$

## Generating Functions

$c$: Profile

$\mathscr{C}_c$: Set of cylindric partitions of profile $c$

$$F_c(z,q) = \sum_{\pi \in \mathscr{C}_c} z^{\max(\pi)} q^{\text{weight}(\pi)}$$

## Generating Functions

$c$: Profile

$\mathscr{C}_c$: Set of cylindric partitions of profile $c$

$$F_c(z,q) = \sum_{\pi \in \mathscr{C}_c} z^{\max(\pi)} q^{\mathrm{weight}(\pi)}$$

$$H_c(z,q) = \frac{(zq;q)_\infty}{(q)_\infty} F_c(z,q)$$

## Key Points

## Key Points

### Products (Borodin / Gessel–Krattenthaler)

$$H_c(1,q) = F_c(1,q) = \frac{1}{(q)_\infty}\chi(\Omega(c_0\Lambda_0 + \cdots + c_r\Lambda_r))$$

## Key Points

### Products (Borodin / Gessel–Krattenthaler)

$$H_c(1,q) = F_c(1,q) = \frac{1}{(q)_\infty}\chi(\Omega(c_0\Lambda_0 + \cdots + c_r\Lambda_r))$$

### Symmetries

$$H_{c_0,c_1,\cdots,c_r}(z,q) = H_{c_1,c_2,\cdots,c_r,c_0}(z,q), \quad H_{c_0,c_1,\cdots,c_r}(1,q) = H_{c_r,c_{r-1},\cdots,c_1,c_0}(1,q)$$

# Key Points

## Products (Borodin / Gessel–Krattenthaler)

$$H_c(1,q) = F_c(1,q) = \frac{1}{(q)_\infty}\chi(\Omega(c_0\Lambda_0 + \cdots + c_r\Lambda_r))$$

## Symmetries

$$H_{c_0,c_1,\cdots,c_r}(z,q) = H_{c_1,c_2,\cdots,c_r,c_0}(z,q), \quad H_{c_0,c_1,\cdots,c_r}(1,q) = H_{c_r,c_{r-1},\cdots,c_1,c_0}(1,q)$$

## Recurrences (Corteel–Welsh)

$H$ functions for all profiles of a fixed rank and level are unique solutions to the Corteel–Welsh system of $z, q$ difference equations.

## With only cyclic symmetry...

## With only cyclic symmetry...

Level 11

## With only cyclic symmetry...

Level 11

| | | | |
|---|---|---|---|
| 11, 0, 0 | | | |
| 10, 1, 0/10, 0, 1 | 9, 1, 1 | | |
| 9, 2, 0/9, 0, 2 | 8, 2, 1/8, 1, 2 | 7, 2, 2 | |
| 8, 3, 0/8, 0, 3 | 7, 3, 1/7, 1, 3 | 6, 3, 2/6, 2, 3 | 5, 3, 3 |
| 7, 4, 0/7, 0, 4 | 6, 4, 1/6, 4, 1 | 5, 4, 2/5, 2, 4 | 4, 4, 3 |
| 6, 5, 0/6, 0, 5 | 5, 5, 1 | | |

# Completing the ASW Ids

# Main Results

## Main Results

**Conjecture (K.–Russell 2022)**

We have explicit (conjectural) $z, q$ Andrews–Schilling–Warnaar sum expressions for *all* $H_c(z, q)$ functions where $c$ is a (length 3) composition "above the line".

# Main Results

**Conjecture (K.–Russell 2022)**

We have explicit (conjectural) $z, q$ Andrews–Schilling–Warnaar sum expressions for *all* $H_c(z, q)$ functions where $c$ is a (length 3) composition “above the line”.

**Theorem (K.–Russell 2022)**

The functions $H_c(z, q)$ where $c$ is a length 3 composition “below the line” are completely determined by the $H_c(z, q)$ functions that lie “above the line”.

## Main Results

**Conjecture (K.–Russell 2022)**

We have explicit (conjectural) $z, q$ Andrews–Schilling–Warnaar sum expressions for *all* $H_c(z,q)$ functions where $c$ is a (length 3) composition "above the line".

**Theorem (K.–Russell 2022)**

The functions $H_c(z,q)$ where $c$ is a length 3 composition "below the line" are completely determined by the $H_c(z,q)$ functions that lie "above the line".

**Corollary**

If you prove the conjecture, setting $z = 1$ leads to $\sum = \prod$ identities for principal characters of standard $\widehat{\mathfrak{sl}}_3$ modules.

## Example: Level 7

## Example: Level 7

$$H_c(z,q) = \sum_{\substack{r_1 \geq r_2 \geq 0 \\ s_1 \geq s_2 \geq 0}} \frac{z^{r_1} q^{(r_1^2 - r_1 s_1 + s_1^2) + (r_2^2 - r_2 s_2 + s_2^2)}}{(q)_{r_1 - r_2} (q)_{s_1 - s_2} (q)_{r_2} (q)_{s_2} (q)_{r_2 + s_2 + 1}} f_c$$

# Example: Level 7

$$H_c(z,q) = \sum_{\substack{r_1 \geq r_2 \geq 0 \\ s_1 \geq s_2 \geq 0}} \frac{z^{r_1} q^{(r_1^2 - r_1 s_1 + s_1^2) + (r_2^2 - r_2 s_2 + s_2^2)}}{(q)_{r_1 - r_2} (q)_{s_1 - s_2} (q)_{r_2} (q)_{s_2} (q)_{r_2 + s_2 + 1}} f_c$$

$7,0,0$

$6,1,0/6,0,1$ $\quad 5,1,1$

$5,2,0/5,0,2$ $\quad 4,2,1/4,1,2$ $\quad 3,2,2$

---

$4,3,0/4,0,3$ $\quad 3,3,1$

| $c$ | $f_c$ |
|---|---|
| $7,0,0$ | $q^{r_1+r_2+s_1+s_2}$ |
| $6,1,0$ | $q^{r_2+s_1+s_2}$ |
| $5,2,0$ | $q^{s_1+s_2}$ |
| $5,1,1$ | $q^{r_2+s_2} - q^{1+r_1+r_2+s_1+s_2}$ |
| $4,2,1$ | $q^{s_2} - q^{1+r_2+s_1+s_2}$ |
| $3,2,2$ | $1 - q^{1+r_2+s_2}$ |

## Example: Level 7

$$H_c(z,q) = \sum_{\substack{r_1 \geq r_2 \geq 0 \\ s_1 \geq s_2 \geq 0}} \frac{z^{r_1} q^{(r_1^2 - r_1 s_1 + s_1^2) + (r_2^2 - r_2 s_2 + s_2^2)}}{(q)_{r_1 - r_2}(q)_{s_1 - s_2}(q)_{r_2}(q)_{s_2}(q)_{r_2+s_2+1}} f_c$$

| | | |
|---|---|---|
| 7, 0, 0 | | |
| 6, 1, 0/6, 0, 1 | 5, 1, 1 | |
| 5, 2, 0/5, 0, 2 | 4, 2, 1/4, 1, 2 | 3, 2, 2 |
| 4, 3, 0/4, 0, 3 | 3, 3, 1 | |

| $c$ | $f_c$ |
|---|---|
| 6, 0, 1 | $q^{r_1+r_2+s_2}$ $-q(1-z)q^{2r_1+r_2+s_1+s_2}$ |
| 5, 0, 2 | $q^{r_1+r_2}$ $-q(1-z)q^{2r_1+r_2+s_2}$ |
| 4, 1, 2 | $q^{r_1} - q^{1+r_1+r_2+s_2}$ |

## Example: Level 7

$$H_c(z,q) = \sum_{\substack{r_1 \geq r_2 \geq 0 \\ s_1 \geq s_2 \geq 0}} \frac{z^{r_1} q^{(r_1^2 - r_1 s_1 + s_1^2) + (r_2^2 - r_2 s_2 + s_2^2)}}{(q)_{r_1 - r_2} (q)_{s_1 - s_2} (q)_{r_2} (q)_{s_2} (q)_{r_2 + s_2 + 1}} f_c$$

7, 0, 0

6, 1, 0/6, 0, 1    5, 1, 1

5, 2, 0/5, 0, 2    4, 2, 1/4, 1, 2    3, 2, 2

---

4, 3, 0/4, 0, 3    3, 3, 1

| $c$ | $f_c$ |
|---|---|
| 4, 3, 0 | $q^{-r_1+s_1+s_2} - q^{r_2+s_2}$ <br> $+(1-z)q^{r_1+s_2}$ <br> $+zq^{1+r_1+r_2+s_1+s_2}$ |
| 4, 0, 3 | $q^{s_2} - q^{1+r_2+s_1+s_2}$ <br> $-zq^{1+r_1+s_1+s_2} - q^{r_1+r_2}$ <br> $(1-qz)q^{r_2} + zq^{2+2r_1+r_2+s_2}$ <br> $zq^{1+2r_1+r_2}$ |

## Example: Level 7

$$H_c(z,q) = \sum_{\substack{r_1 \geq r_2 \geq 0 \\ s_1 \geq s_2 \geq 0}} \frac{z^{r_1} q^{(r_1^2 - r_1 s_1 + s_1^2) + (r_2^2 - r_2 s_2 + s_2^2)}}{(q)_{r_1 - r_2} (q)_{s_1 - s_2} (q)_{r_2} (q)_{s_2} (q)_{r_2 + s_2 + 1}} f_c$$

| | | |
|---|---|---|
| $7,0,0$ | | |
| $6,1,0/6,0,1$ | $5,1,1$ | |
| $5,2,0/5,0,2$ | $4,2,1/4,1,2$ | $3,2,2$ |
| $4,3,0/4,0,3$ | $3,3,1$ | |

| $c$ | $f_c$ |
|---|---|
| $3,3,1$ | $q^{-r_1+s_2} + (1-z)q^{r_1}$ $-q^{r_2} + zq^{1+r_1+r_2+s_2}$ $-q^{1-r_1+r_2+s_1+s_2} - zq^{s_1+s_2}$ |

# Main Results

## Main Results

**Theorem (K.–Russell 2022)**

Our conjectures hold when $c_0 + c_1 + c_2 = \text{level} \in \{2, \mathbf{3}, 4, 5, \mathbf{7}\}$.

## Main Results

**Theorem (K.–Russell 2022)**

Our conjectures hold when $c_0 + c_1 + c_2 =$ level $\in \{2, \mathbf{3}, 4, 5, \mathbf{7}\}$.

Proof: Show explicitly and computationally that our conjectures satisfy the required Corteel–Welsh recursions at these levels.

## Main Results

**Theorem (K.–Russell 2022)**

Our conjectures hold when $c_0 + c_1 + c_2 =$ level $\in \{2, \mathbf{3}, 4, 5, \mathbf{7}\}$.

Proof: Show explicitly and computationally that our conjectures satisfy the required Corteel–Welsh recursions at these levels.

**Note**

Smallest unproved level is $6$. (Levels divisible by $3$ are funny).

## Recent Results

## Recent Results

**Theorem (Warnaar 2023)**

The $\sum = \prod$ conjectures obtained by setting $z \mapsto 1$ hold for all levels for compositions that are above the line.

## Recent Results

**Theorem (Warnaar 2023)**

The $\sum = \prod$ conjectures obtained by setting $z \mapsto 1$ hold for all levels for compositions that are above the line.

**Note**

Our bivariate conjectures are widely open in general (smallest unproved level is $6$).

# Torus Knots

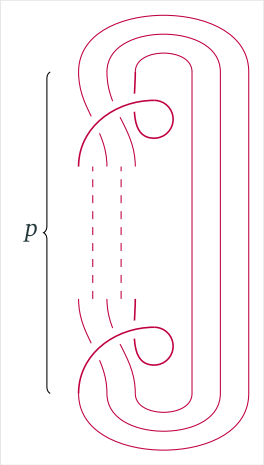
# Torus Knots $(3, p)$

# Torus Knots $(3, p)$

Consider the Torus Knots $T(3, p)$ with $3 \nmid p$. [a]

Colour with irreducible module $L_3(n\Lambda_1)$ of $\mathfrak{sl}_3$.

Consider the coloured Jones invariants: $J_{T(3,p)}(L_3(n\Lambda_1))$.




[a] remember that thing about levels divisible by 3 being funny?

# Limits of coloured invariants

## Limits of coloured invariants

**Theorem (K. 2023)**

We have the following limit for $p > 3$, $3 \nmid p$:

$$\lim_{n \to \infty} J_{T(3,p)}(L_3(n\Lambda_1)) = \frac{(q)_\infty^2}{(1-q)^2(1-q^2)} \chi(\Omega((p-3)\Lambda_0)).$$

## Limits of coloured invariants

**Theorem (K. 2023)**

We have the following limit for $p > 3$, $3 \nmid p$:

$$\lim_{\mathbf{n}\to\infty} J_{T(3,p)}(L_3(\mathbf{n}\Lambda_1)) = \frac{(q)_\infty^2}{(1-q)^2(1-q^2)}\chi(\Omega((p-3)\Lambda_0)).$$

**Note**

The theorem is valid much more generally for characters of all minimal model principal $\mathscr{W}$ algebras of type $\mathrm{A}_r$: $\mathscr{W}_r(p,p')$.

Question

## Question

**Question**

What can knot-theoretic methods say about the Andrews–Schilling–Warnaar sums or cylindric partitions?

Thank you!