On the $A_2$ Andrews–Schilling–Warnaar Identities

Shashank Kanade
University of Denver

Based on joint work with Matthew C. Russell
Cast and the Characters
Rogers–Ramanujan identities

\[
\begin{align*}
\mathcal{P}_n \left( q^n \right) &= 1 - \left( q, q^4 \right) \left( q^5 \right) \\
\mathcal{Q}_n \left( q^n \right) &= 1 - \left( q^2, q^3 \right) \left( q^5 \right)
\end{align*}
\]

VOAs, affine Lie algebras, and related structures

Knot Theory
Rogers–Ramanujan identities

\[
\sum_{n \geq 0} \frac{q^{n^2}}{(q)_n} = \frac{1}{(q, q^4; q^5)_\infty}
\]

\[
\sum_{n \geq 0} \frac{q^{n^2+n}}{(q)_n} = \frac{1}{(q^2, q^3; q^5)_\infty}
\]
Rogers–Ramanujan identities

\[
\sum_{n \geq 0} \frac{q^{n^2}}{(q)_n} = \frac{1}{(q, q^4; q^5)_\infty}
\]

\[
\sum_{n \geq 0} \frac{q^{n^2+n}}{(q)_n} = \frac{1}{(q^2, q^3; q^5)_\infty}
\]

VOAs, affine Lie algebras, and related structures
Rogers–Ramanujan identities

\[ \sum_{n \geq 0} \frac{q^{n^2}}{(q)_n} = \frac{1}{(q, q^4; q^5)_\infty} \]

\[ \sum_{n \geq 0} \frac{q^{n^2+n}}{(q)_n} = \frac{1}{(q^2, q^3; q^5)_\infty} \]

VOAs, affine Lie algebras, and related structures

Knot Theory
$A_2$ Andrews–Schilling–Warnaar identities

$\sum = \prod$

Sums by Andrews–Schilling–Warnaar

$A_2$ Bailey

Principal Characters of Standard Modules of $\hat{sl}_3$

$(z, q)$ Sums $\leftrightarrow$ Cylindric partitions

$z = 1$ Product by Borodin or Gessel–Krattenthaler
Setup
Principal Characters: $\hat{5|}_3$
Principal Characters: $\tilde{\mathfrak{sl}}_3$

$$\lambda = c_0 \Lambda_0 + c_1 \Lambda_1 + c_2 \Lambda_2 \ (c_i \in \mathbb{Z}_{\geq 0})$$
Principal Characters: $\tilde{\mathfrak{sl}}_3$

$$\lambda = c_0 \Lambda_0 + c_1 \Lambda_1 + c_2 \Lambda_2 \ (c_i \in \mathbb{Z}_{\geq 0})$$

Level = $\ell = c_0 + c_1 + c_2$
Principal Characters: $\hat{\mathfrak{sl}}_3$

\[ \lambda = c_0 \Lambda_0 + c_1 \Lambda_1 + c_2 \Lambda_2 \ (c_i \in \mathbb{Z}_{\geq 0}) \]

Level = $\ell = c_0 + c_1 + c_2$

Modulus = $m = \ell + 3 = c_0 + c_1 + c_2 + 3$
Principal Characters: $\tilde{sl}_3$

\[ \lambda = c_0 \Lambda_0 + c_1 \Lambda_1 + c_2 \Lambda_2 \quad (c_i \in \mathbb{Z}_{\geq 0}) \]

Level = $\ell = c_0 + c_1 + c_2$

Modulus = $m = \ell + 3 = c_0 + c_1 + c_2 + 3$

Principally specialized character:

\[ \chi(L(\lambda)) = \left( e^{-\lambda} \text{ch}(L(\lambda)) \right) |_{e^{-a_i} \rightarrow q} \]
Principal Characters: $\mathfrak{sl}_3$

$\lambda = c_0 \Lambda_0 + c_1 \Lambda_1 + c_2 \Lambda_2 \ (c_i \in \mathbb{Z}_{\geq 0})$

Level $= \ell = c_0 + c_1 + c_2$

Modulus $= m = \ell + 3 = c_0 + c_1 + c_2 + 3$

Principally specialized character:

$\chi(L(\lambda)) = (e^{-\lambda} \text{ch}(L(\lambda))) \big|_{e^{-\alpha_i} \rightarrow q}$

Principal character

$\chi(\Omega(\lambda)) = \frac{\chi(L(\lambda))}{\chi(L(\Lambda_0))}$
Principal Characters:\ \( \tilde{\mathfrak{sl}}_3 \)

\[\begin{align*}
\lambda &= c_0 \Lambda_0 + c_1 \Lambda_1 + c_2 \Lambda_2 \quad (c_i \in \mathbb{Z}_{\geq 0}) \\
\text{Level} &= \ell = c_0 + c_1 + c_2 \\
\text{Modulus} &= m = \ell + 3 = c_0 + c_1 + c_2 + 3 \\
\text{Principally specialized character:} \\
\chi(L(\lambda)) &= \left( e^{-\lambda} \, \text{ch}(L(\lambda)) \right) \big|_{e^{-a_i} \to q}
\end{align*}\]

<table>
<thead>
<tr>
<th>Principal character</th>
</tr>
</thead>
</table>
|\[\begin{align*}
\chi(\Omega(\lambda)) &= \frac{\chi(L(\lambda))}{\chi(L(\Lambda_0))} \\
&= \frac{(q^m; q^m)_\infty^2}{(q)^2_\infty} (q^{1+c_0}, q^{m-1-c_0}, q^{1+c_1}, q^{m-1-c_1}, q^{1+c_2}, q^{m-1-c_2}; q^m)_\infty
\end{align*}\]|

3
Arrangement of Modules
Arrangement of Modules

Level 11
### Arrangement of Modules

#### Level 11

<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>11,0,0</td>
<td>10,1,0</td>
<td>9,1,1</td>
<td>9,2,0</td>
<td>8,2,1</td>
<td>7,2,2</td>
</tr>
<tr>
<td>8,3,0</td>
<td>7,3,1</td>
<td>6,3,2</td>
<td>5,3,3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7,4,0</td>
<td>6,4,1</td>
<td>5,4,2</td>
<td>4,4,3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6,5,0</td>
<td>5,5,1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Sample identities (ASW)
Sample identities (ASW)

Level 7 (Andrews–Schilling–Warnaar)

\[
\sum_{r_1 \geq r_2 \geq 0 \atop s_1 \geq s_2 \geq 0} \frac{q^{(r_1^2 - r_1 s_1 + s_1^2) + (r_2^2 - r_2 s_2 + s_2^2) + r_1 + r_2 + s_1 + s_2}}{(q)_{r_1 - r_2} (q)_{s_1 - s_2} (q)_{r_2} (q)_{s_2} (q)_{r_2 + s_2 + 1}} = \frac{1}{(q)_{\infty}} \chi(\Omega(7\Lambda_0 + 0\Lambda_1 + 0\Lambda_2))
\]

\[
= \frac{(q^{10}; q^{10})_{\infty}^2}{(q)^3_{\infty}} (q, q, q^2, q^8, q^9, q^9; q^{10})_{\infty}
\]
Sample identities (ASW)

Level 5 (Andrews–Schilling–Warnaar)

\[
\sum_{r_1 \geq r_2 \geq 0 \atop s_1 \geq s_2 \geq 0} \frac{q^{(r_1^2 - r_1 s_1 + s_1^2) + (r_2^2 + r_2 s_2 + s_2^2)} + r_1 + r_2 + s_1 + s_2}{(q)_{r_1 - r_2} (q)_{s_1 - s_2} (q)_{r_2} (q)_{s_2} (q)_{r_2 + s_2 + 1}}
\]

\[= \frac{1}{(q)_{\infty}} \chi(\Omega(5\Lambda_0 + 0\Lambda_1 + 0\Lambda_2)) \]

\[= \frac{(q^8; q^8)_{\infty}^2}{(q)_{\infty}^3} (q, q, q^2, q^6, q^7, q^7; q^8)_{\infty} \]
$\sum_{r_1 \geq r_2 \geq 0, s_1 \geq s_2 \geq 0} q^{(r_1^2 - r_1 s_1 + s_1^2) + (r_2^2 - r_2 s_2 + s_2^2) + r_1 + r_2 + s_1 + s_2} \frac{(q)_{r_1 - r_2} (q)_{s_1 - s_2} (q)^{r_2 + s_2} (q)_{r_2 + s_2 + 1}}{(q)_{r_2} \chi((q)_{6\Lambda_0 + 0\Lambda_1 + 0\Lambda_2})^2 (q^9; q^9)^2 (q^9; q^9)_{\infty}}$

$= \frac{1}{(q)_{\infty}}$
The main question

Find and prove ASW identities for the remaining principal characters of \( b_{s_l} \).
The main question

Question
Find and prove ASW identities for the remaining principal characters of $\tilde{sl}_3$. 
Cylindric Partitions
Plane Partitions
Plane Partitions

A plane partition
A plane partition

<table>
<thead>
<tr>
<th>7</th>
<th>4</th>
<th>3</th>
<th>2</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>4</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
A *plane partition*

<table>
<thead>
<tr>
<th>7</th>
<th>4</th>
<th>3</th>
<th>2</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>4</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Numbers in boxes as heights
A plane partition

Numbers in boxes as heights → 3-dimensional shape
A plane partition

Numbers in boxes as heights $\rightarrow$ 3-dimensional shape
Cylindric Partitions
Cylindric Partitions

Cylindric partition of profile \( c = (3, 1, 2) \)
Cylindric Partitions

Cylindric partition of profile $c = (3, 1, 2)$

Connected to $\chi(\Omega(3\Lambda_0 + 1\Lambda_1 + 2\Lambda_2))$ for $\hat{s}_{l_3} = A_2^{(1)}$
Cylindric Partitions

Cylindric partition of profile $c = (3, 1, 2)$

Connected to $\chi(\Omega(3\Lambda_0 + 1\Lambda_1 + 2\Lambda_2))$ for $\mathfrak{s}l_3 = A_2^{(1)}$
Cylindric Partitions

Cylindric partition of profile $c = (3, 1, 2)$

Connected to $\chi(\Omega(3 \Lambda_0 + 1 \Lambda_1 + 2 \Lambda_2))$ for $\hat{s}_3 = A^{(1)}_2$

Total “skew-ness” = level = $3 + 1 + 2 = 6$

Weight = $5 + 5 + 4 + 3 + 6 + 5 + 3 + 7 + 6 + 2$

Max-part = 7
Generating Functions

\[ \text{Generating Functions} \]
Generating Functions

c: Profile

\[ F_c(z, q) = \frac{P_{\pi_2}^{C_c}}{z_{\text{max}}(\pi)} q^{\text{weight}(\pi)} \]

\[ H_c(z, q) = (zq; q)_1 \]

\[ F_c(z, q) \]
Generating Functions

c: Profile

$\mathcal{C}_c$: Set of cylindric partitions of profile $c$
Generating Functions

c: Profile

\( \mathcal{C}_c: \text{Set of cylindric partitions of profile } c \)

\[
F_c(z, q) = \sum_{\pi \in \mathcal{C}_c} z^{\text{max}(\pi)} q^{\text{weight}(\pi)}
\]
Generating Functions

c: Profile

$\mathcal{C}_c$: Set of cylindric partitions of profile $c$

$$F_c(z, q) = \sum_{\pi \in \mathcal{C}_c} z^{\max(\pi)} q^{\text{weight}(\pi)}$$

$$H_c(z, q) = \frac{(zq; q)_\infty}{(q)_\infty} F_c(z, q)$$
Key Points

Products (Borodin / Gessel–Krattenthaler)

\[ H_c(1, q) = F_c(1, q) = 1 (q) \chi(\Omega(c_0\Lambda_0 + \cdots + c_r\Lambda_r)) \]

Symmetries

\[ H_{c_0}, c_1, \ldots, c_r(z, q) = H_{c_1}, c_2, \ldots, c_r(z, q) = H_{c_0}, c_1, \ldots, c_r(1, q) = H_{c_r}, \ldots, c_r(1, q) \]

Recurrences (Corteel–Welsh)

\[ \text{H functions for all profiles of a fixed rank and level are unique solutions to the Corteel–Welsh system of difference equations.} \]
Key Points

Products (Borodin / Gessel–Krattenthaler)

\[ H_c(1, q) = F_c(1, q) = \frac{1}{(q)_\infty} \chi(\Omega(c_0 \Lambda_0 + \cdots + c_r \Lambda_r)) \]
Key Points

Products (Borodin / Gessel–Krattenthaler)

\[ H_c(1, q) = F_c(1, q) = \frac{1}{(q)_{\infty}} \chi(\Omega(c_0\Lambda_0 + \cdots + c_r\Lambda_r)) \]

Symmetries

\[ H_{c_0, c_1, \ldots, c_r}(z, q) = H_{c_1, c_2, \ldots, c_r, c_0}(z, q), \quad H_{c_0, c_1, \ldots, c_r}(1, q) = H_{c_r, c_{r-1}, \ldots, c_1, c_0}(1, q) \]
Key Points

Products (Borodin / Gessel–Krattenthaler)

\[ H_c(1, q) = F_c(1, q) = \frac{1}{(q)_\infty} \chi(\Omega(c_0\Lambda_0 + \cdots + c_r\Lambda_r)) \]

Symmetries

\[ H_{c_0, c_1, \ldots, c_r}(z, q) = H_{c_1, c_2, \ldots, c_r, c_0}(z, q), \quad H_{c_0, c_1, \ldots, c_r}(1, q) = H_{c_r, c_{r-1}, \ldots, c_1, c_0}(1, q) \]

Recurrences (Corteel–Welsh)

\( H \) functions for all profiles of a fixed rank and level are unique solutions to the Corteel–Welsh system of \( z, q \) difference equations.
With only cyclic symmetry...
With only cyclic symmetry...

Level 11
With only cyclic symmetry...

**Level 11**

<table>
<thead>
<tr>
<th>11, 0, 0</th>
<th>10, 1, 0/10, 0, 1</th>
<th>9, 1, 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>9, 2, 0/9, 0, 2</td>
<td>8, 2, 1/8, 1, 2</td>
<td>7, 2, 2</td>
</tr>
<tr>
<td>8, 3, 0/8, 0, 3</td>
<td>7, 3, 1/7, 1, 3</td>
<td>6, 3, 2/6, 2, 3</td>
</tr>
<tr>
<td>7, 4, 0/7, 0, 4</td>
<td>6, 4, 1/6, 4, 1</td>
<td>5, 4, 2/5, 2, 4</td>
</tr>
<tr>
<td>6, 5, 0/6, 0, 5</td>
<td></td>
<td>5, 5, 1</td>
</tr>
</tbody>
</table>
Completing the ASW Ids
Main Results

Conjecture (K.–Russell 2022)
We have explicit (conjectural) Andrews–Schilling–Warnaar sum expressions for all $H_c(z, q)$ functions where $c$ is a (length 3) composition “above the line”.

Theorem (K.–Russell 2022)
The functions $H_c(z, q)$ where $c$ is a length 3 composition “below the line” are completely determined by the $H_c(z, q)$ functions that lie “above the line”.

Corollary
If you prove the conjecture, setting $z = 1$ leads to identities for principal characters of standard $b_{s3}$ modules.
Main Results

Conjecture (K.–Russell 2022)
We have explicit (conjectural) $z, q$ Andrews–Schilling–Warnaar sum expressions for all $H_c(z, q)$ functions where $c$ is a (length 3) composition “above the line”.

Theorem (K.–Russell 2022)
The functions $H_c(z, q)$ where $c$ is a length 3 composition “below the line” are completely determined by the $H_c(z, q)$ functions that lie “above the line”.

Corollary
If you prove the conjecture, setting $z = 1$ leads to $P = Q$ identities for principal characters of standard $b_{sl_3}$ modules.
Main Results

Conjecture (K.–Russell 2022)
We have explicit (conjectural) $z, q$ Andrews–Schilling–Warnaar sum expressions for all $H_c(z, q)$ functions where $c$ is a (length 3) composition “above the line”.

Theorem (K.–Russell 2022)
The functions $H_c(z, q)$ where $c$ is a length 3 composition “below the line” are completely determined by the $H_c(z, q)$ functions that lie “above the line”.

Corollary
If you prove the conjecture, setting $z = 1$ leads to $P = Q$ identities for principal characters of standard $\mathfrak{sl}_3$ modules.
**Main Results**

**Conjecture (K.–Russell 2022)**
We have explicit (conjectural) $z, q$ Andrews–Schilling–Warnaar sum expressions for all $H_c(z, q)$ functions where $c$ is a (length 3) composition “above the line”.

**Theorem (K.–Russell 2022)**
The functions $H_c(z, q)$ where $c$ is a length 3 composition “below the line” are completely determined by the $H_c(z, q)$ functions that lie “above the line”.

**Corollary**
If you prove the conjecture, setting $z = 1$ leads to $\sum = \prod$ identities for principal characters of standard $\hat{sl}_3$ modules.
Example: Level 7
Example: Level 7

\[ H_c(z, q) = \sum_{r_1 \geq r_2 \geq 0, s_1 \geq s_2 \geq 0} \frac{z^{r_1} q^{(r_1^2 - r_1 s_1 + s_1^2) + (r_2^2 - r_2 s_2 + s_2^2)}}{(q)_{r_1 - r_2} (q)_{s_1 - s_2} (q)_{r_2} (q)_{s_2} (q)_{r_2 + s_2 + 1}} f_c \]
Example: Level 7

\[ H_c(z, q) = \sum_{\substack{r_1 \geq r_2 \geq 0 \ \mathrm{and} \ s_1 \geq s_2 \geq 0}} \frac{z^{r_1} q^{(r_1^2 - r_1 s_1 + s_1^2) + (r_2^2 - r_2 s_2 + s_2^2)}}{(q)_1 (q)_2 (q)_s_1 (q)_r_1 (q)_s_2 (q)_r_2 + s_2 + 1} f_c \]

<table>
<thead>
<tr>
<th>c</th>
<th>( f_c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>7, 0, 0</td>
<td>( q^{r_1 + r_2 + s_1 + s_2} )</td>
</tr>
<tr>
<td>6, 1, 0/6, 0, 1</td>
<td>( q^{r_2 + s_2} )</td>
</tr>
<tr>
<td>5, 2, 0/5, 0, 2</td>
<td>( q^{s_1 + s_2} )</td>
</tr>
<tr>
<td>4, 2, 1/4, 1, 2</td>
<td>( q^{r_1 + r_2 + s_1 + s_2} - q^{1 + r_1 + r_2 + s_1 + s_2} )</td>
</tr>
<tr>
<td>3, 2, 2</td>
<td>( q^{s_2} - q^{1 + r_2 + s_1 + s_2} )</td>
</tr>
<tr>
<td>3, 3, 1</td>
<td>( 1 - q^{1 + r_2 + s_2} )</td>
</tr>
</tbody>
</table>
Example: Level 7

\[ H_c(z, q) = \sum_{r_1 \geq r_2 \geq 0 \atop s_1 \geq s_2 \geq 0} z^{r_1} q^{(r_1^2 - r_1 s_1 + s_1^2) + (r_2^2 - r_2 s_2 + s_2^2)} \frac{(q)_{r_1 - r_2} (q)_{s_1 - s_2} (q)_{r_2} (q)_{s_2}}{(q)_{r_2 + s_2 + 1}} f_c \]

<table>
<thead>
<tr>
<th>c</th>
<th>f_c</th>
</tr>
</thead>
<tbody>
<tr>
<td>7, 0, 0</td>
<td></td>
</tr>
<tr>
<td>6, 1, 0</td>
<td>6, 0, 1</td>
</tr>
<tr>
<td></td>
<td>5, 1, 1</td>
</tr>
<tr>
<td>5, 2, 0</td>
<td>5, 0, 2</td>
</tr>
<tr>
<td></td>
<td>4, 2, 1/4, 1, 2</td>
</tr>
<tr>
<td></td>
<td>3, 2, 2</td>
</tr>
<tr>
<td>4, 3, 0</td>
<td>4, 0, 3</td>
</tr>
<tr>
<td></td>
<td>3, 3, 1</td>
</tr>
</tbody>
</table>

\[ -q(1 - z) q^{2r_1 + r_2 + s_1 + s_2} \]

<table>
<thead>
<tr>
<th>6, 0, 1</th>
<th>( q^{r_1 + r_2 + s_2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5, 0, 2</td>
<td>( q^{r_1 + r_2} )</td>
</tr>
<tr>
<td></td>
<td>( -q(1 - z) q^{2r_1 + r_2 + s_2} )</td>
</tr>
<tr>
<td>4, 1, 2</td>
<td>( q^{r_1} - q^{1 + r_1 + r_2 + s_2} )</td>
</tr>
</tbody>
</table>
Example: Level 7

\[ H_c(z, q) = \sum_{r_1 \geq r_2 \geq 0 \atop s_1 \geq s_2 \geq 0} z^{r_1} q^{(r_1^2 - r_1 s_1 + s_1^2) + (r_2^2 - r_2 s_2 + s_2^2)} \frac{(q^{r_1 - r_2})(q^{s_1 - s_2})(q^{r_2})(q^{s_2})}{s_1 + s_2 + 1} f_c \]

<table>
<thead>
<tr>
<th>(c)</th>
<th>(f_c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4, 3, 0</td>
<td>(q^{-r_1 + s_1 + s_2} - q^{r_2 + s_2})</td>
</tr>
<tr>
<td></td>
<td>(+(1 - z)q^{r_1 + s_2})</td>
</tr>
<tr>
<td></td>
<td>(+zq^{1 + r_1 + r_2 + s_1 + s_2})</td>
</tr>
<tr>
<td>4, 0, 3</td>
<td>(q^{s_2} - q^{1 + r_2 + s_1 + s_2})</td>
</tr>
<tr>
<td></td>
<td>(-zq^{1 + r_1 + s_1 + s_2} - q^{r_1 + r_2})</td>
</tr>
<tr>
<td></td>
<td>((1 - qz)q^{r_2} + zq^{2 + 2r_1 + r_2 + s_2})</td>
</tr>
<tr>
<td></td>
<td>(zq^{1 + 2r_1 + r_2})</td>
</tr>
</tbody>
</table>
Example: Level 7

\[ H_c(z, q) = \sum_{\substack{r_1 \geq r_2 \geq 0 \\ \ \ s_1 \geq s_2 \geq 0}} \frac{z^{r_1} q^{(r_1^2 - r_1 s_1 + s_1^2) + (r_2^2 - r_2 s_2 + s_2^2)}}{(q)_{r_1 - r_2} (q)_{s_1 - s_2} (q)_{r_2} (q)_{s_2} (q)_{r_2 + s_2 + 1}} f_c \]

<table>
<thead>
<tr>
<th>c</th>
<th>f_c</th>
</tr>
</thead>
<tbody>
<tr>
<td>3, 3, 1</td>
<td>( q^{-r_1 + s_2} + (1 - z) q^{r_1} )</td>
</tr>
<tr>
<td></td>
<td>(-q^{r_2} + z q^{1 + r_1 + r_2 + s_2} )</td>
</tr>
<tr>
<td></td>
<td>(-q^{1 - r_1 + r_2 + s_1 + s_2} - z q^{s_1 + s_2} )</td>
</tr>
</tbody>
</table>
Main Results

Theorem (K.–Russell 2022)

Our conjectures hold when \( c_0 + c_1 + c_2 = \) level \( 2, 3, 4, 5, 7 \).

Proof: Show explicitly and computationally that our conjectures satisfy the required Corteel–Welsh recursions at these levels.

Note: Smallest unproved level is 6. (Levels divisible by 3 are funny).
Main Results

**Theorem (K.–Russell 2022)**
Our conjectures hold when $c_0 + c_1 + c_2 = \text{level} \in \{2, 3, 4, 5, 7\}$. 

Proof: Show explicitly and computationally that our conjectures satisfy the required Corteel–Welsh recursions at these levels.

Note Smallest unproved level is 6. (Levels divisible by 3 are funny).
Main Results

Theorem (K.–Russell 2022)

Our conjectures hold when $c_0 + c_1 + c_2 = \text{level} \in \{2, 3, 4, 5, 7\}$.

Proof: Show explicitly and computationally that our conjectures satisfy the required Corteel–Welsh recursions at these levels.
Main Results

Theorem (K.–Russell 2022)

Our conjectures hold when $c_0 + c_1 + c_2 = \text{level} \in \{2, 3, 4, 5, 7\}$.

Proof: Show explicitly and computationally that our conjectures satisfy the required Corteel–Welsh recursions at these levels.

Note

Smallest unproved level is 6. (Levels divisible by 3 are funny).
Recent Results

Theorem (Warnaar 2023)

The $P = Q$ conjectures obtained by setting $z_7! = 1$ hold for all levels for compositions that are above the line.

Note: Our bivariate conjectures are widely open in general (smallest unproved level is 6).
Recent Results

**Theorem (Warnaar 2023)**

The $\sum = \prod$ conjectures obtained by setting $z \mapsto 1$ hold for all levels for compositions that are above the line.
Recent Results

Theorem (Warnaar 2023)

The $\sum = \prod$ conjectures obtained by setting $z \rightarrow 1$ hold for all levels for compositions that are above the line.

Note

Our bivariate conjectures are widely open in general (smallest unproved level is 6).
Torus Knots
Consider the Torus Knots $T(3,p)$ with $3 \nmid p$.

Colour with irreducible module $L_3(n \Lambda_1)$ of $sl_3$.

Consider the coloured Jones invariants: $J_{T(3,p)}(L_3(n \Lambda_1))$.

Remember that thing about levels divisible by 3 being funny?
Consider the Torus Knots $T(3, p)$ with $3 \nmid p$. \(^a\)

Colour with irreducible module $L_3(n\Lambda_1)$ of $\mathfrak{sl}_3$.

Consider the coloured Jones invariants:

$J_{T(3,p)}(L_3(n\Lambda_1))$.

\(^a\)remember that thing about levels divisible by 3 being funny?
Theorem (K. 2023)

We have the following limit for $p > 3$, $3 

\lim_{n \to 1} J^T(3, p) \left( L_3(n \Lambda_1) \right) = \left( \frac{q}{2} \right)^2 \left( 1 - \frac{q}{2} \right)^2 \chi(\Omega((p^3) \Lambda_0)).$

Note

The theorem is valid much more generally for characters of all minimal model principal $W$-algebras of type $A_r$.

Limits of coloured invariants
Theorem (K. 2023)

We have the following limit for $p > 3$, $3 \nmid p$:

$$\lim_{n \to \infty} J_{T(3,p)}(L_3(n\Lambda_1)) = \frac{(q)_\infty^2}{(1-q)^2(1-q^2)} \chi(\Omega((p-3)\Lambda_0)),$$

where $\chi(\Omega((p-3)\Lambda_0))$ is the character associated with the given representation.
Limits of coloured invariants

Theorem (K. 2023)

We have the following limit for $p > 3$, $3 
mid p$:

$$\lim_{n \to \infty} J_{T(3,p)}(L_3(n \Lambda_1)) = \frac{(q)^2}{(1 - q)^2(1 - q^2)} \chi(\Omega((p - 3)\Lambda_0)).$$

Note

The theorem is valid much more generally for characters of all minimal model principal $\mathcal{W}$ algebras of type $A_r$: $\mathcal{W}_r(p, p')$. 
Question

What can knot-theoretic methods say about the Andrews–Schilling–Warnaar sums or cylindric partitions?
What can knot-theoretic methods say about the Andrews–Schilling–Warnaar sums or cylindric partitions?
Thank you!