From Kreweras to Gessel: A walk through patterns in the quarter plane.

21 July 2023

Formal Power Series and Algebraic Combinatorics 2023

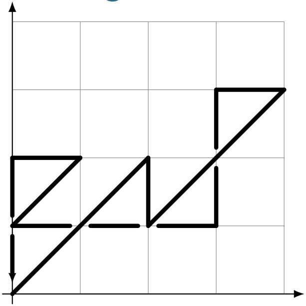
Joint work with: Andrei Asinowski and Cyril Banderier

FWF Der Wissenschaftsfonds.

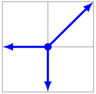
UNIVERSITÄT KLAGENFURT

Modeling - Analysis - Optimization DOC.FUNDS DOCTORAL SCHOOL

Unravelling a formula!



$\mathcal{S} = \{(1,1), (0,-1), (-1,0)\}$.



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Meander: $(0,0)$ to anywhere

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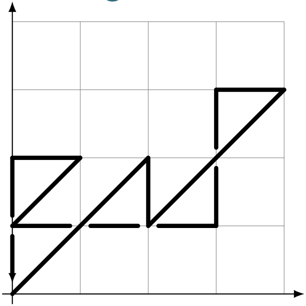

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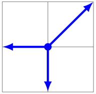

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How many such excursions are there?

$$\frac{4^n}{(n+1)(2n+1)}\binom{3n}{n}$$

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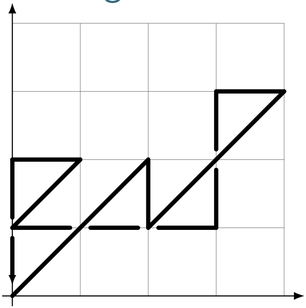

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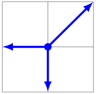

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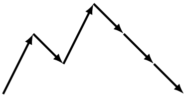

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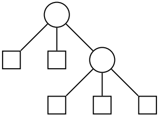

Ternary trees

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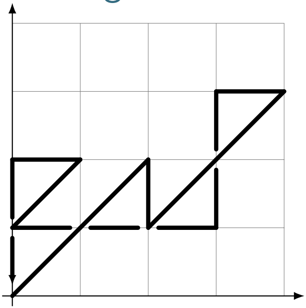

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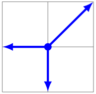

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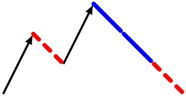

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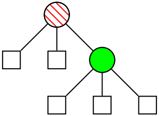
4-colouring



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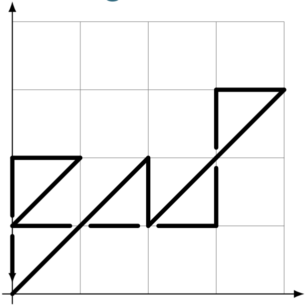

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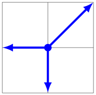

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Equivalence classes!

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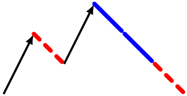

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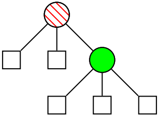
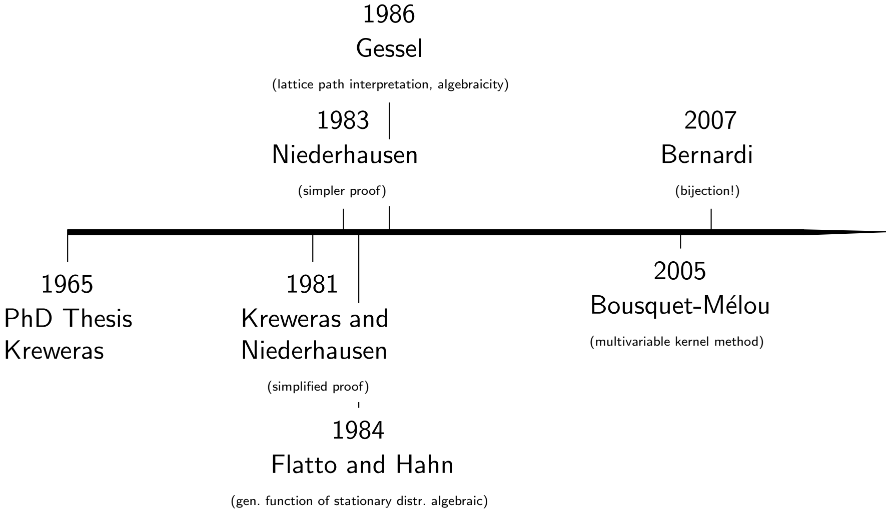


Ternary trees

A brief history of Kreweras walks



A brief history of ~~Kreweras walks~~ walks in the quarter plane

1965
PhD Thesis
Kreweras

1981
Kreweras and
Niederhausen

(simplified proof)

1983
Niederhausen

(simpler proof)

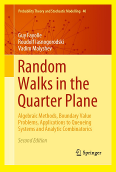
1984
Flatto and Hahn

(gen. function of stationary distr. algebraic)

1986
Gessel

(lattice path interpretation, algebraicity)

1999



2005
Bousquet-Mélou

(multivariable kernel method)

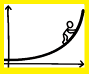
2007
Bernardi

(bijection!)

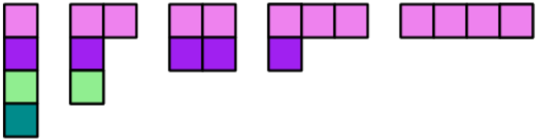
2010
Bousquet-Mélou and
Mishna

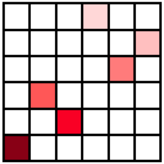
(general classification – "small steps")

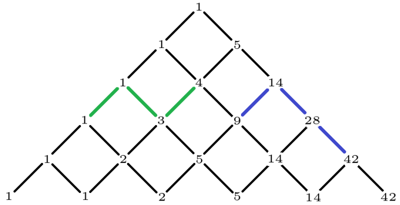
2013



Combinatorial structures where patterns are commonly studied:







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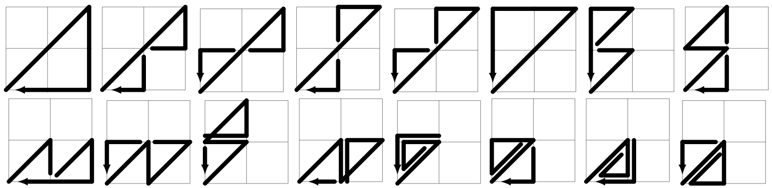
(links with sorting algorithms, logic, number theory, bioinformatics, . . .)

Typical questions:

- What is number of structures of size n with j occurrences of the pattern?
- Is there a nice formula for the generating function?
- Asymptotic behaviour, limit laws?
- Generation of these objects?

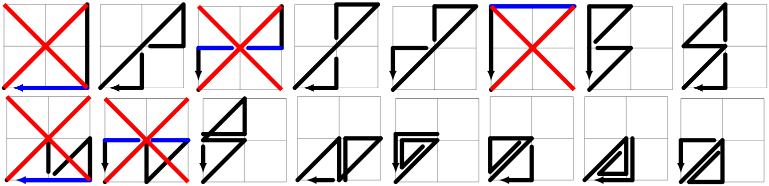
Patterns in walks in the quarter plane: A humble start

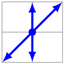
Kreweras excursions of length 6



Patterns in walks in the quarter plane: A humble start

Kreweras excursions of length 6 avoiding the pattern ←←.



1, 2, 11, 85, 782, 8004, . . . OEIS A135404: "Gessel excursions" 

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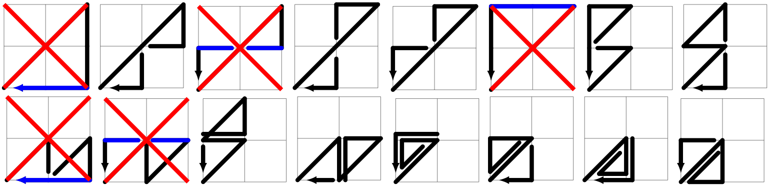

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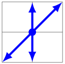
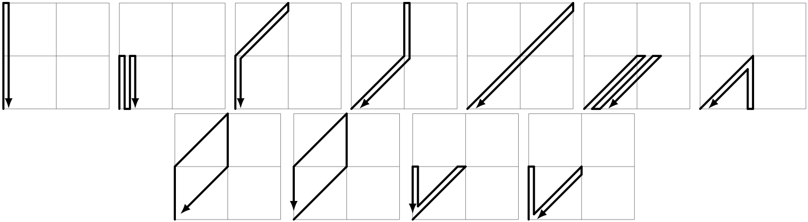
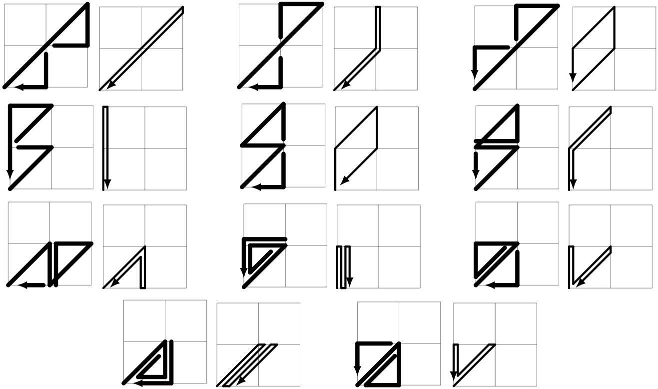


Gessel excursions of length 4.



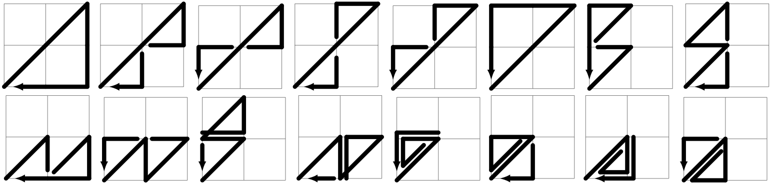
Patterns in walks in the quarter plane: A humble start

‘Shortcut’ bijection (Asinowski, Banderier, S.):



What about other patterns?

Kreweras excursions of length 6



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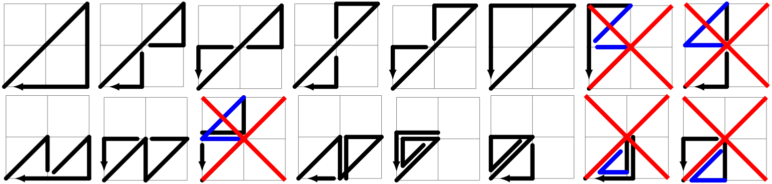
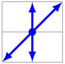


1, 2, 11, 85, 782, 8004, . . . OEIS A135404: "Gessel excursions" 

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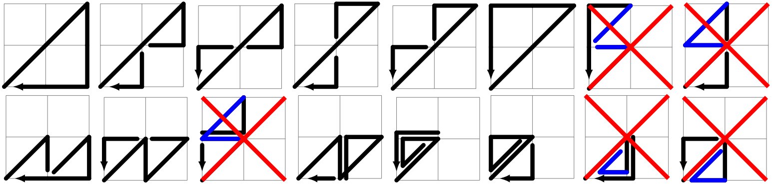
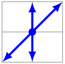
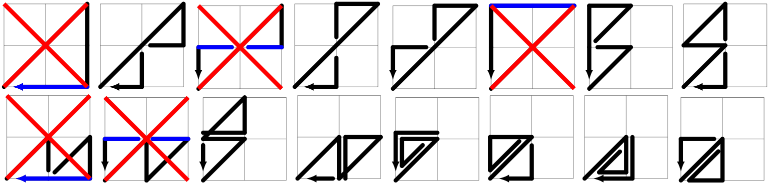


1, 2, 11, 85, 782, 8004, . . . OEIS A135404: "Gessel excursions" 

Kreweras excursions of length 6 avoiding the pattern .



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Proposition (Asinowski, Banderier, S.)

The number of Kreweras excursions of length 3n with

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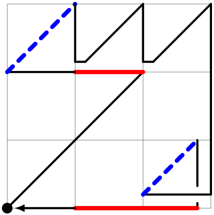

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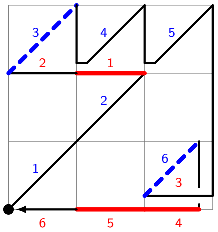

(1, 4, 5)

(3, 6)

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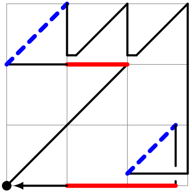

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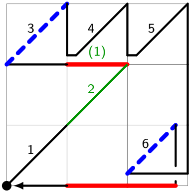

move (1) before 2

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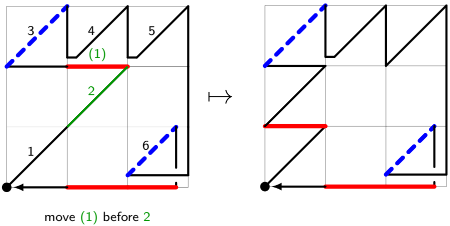

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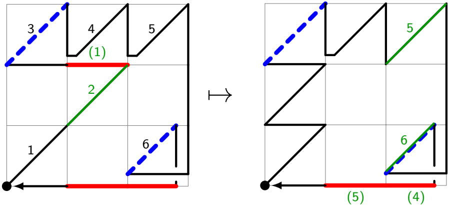

move (1) before 2

move (4) before 5
move (5) before 6

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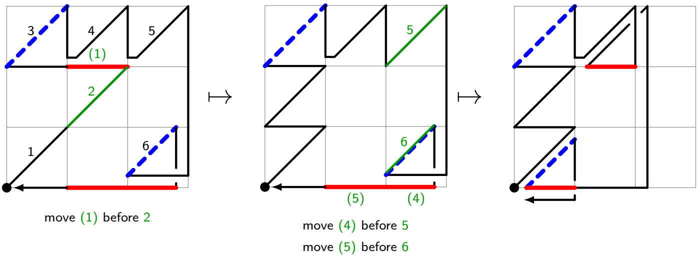

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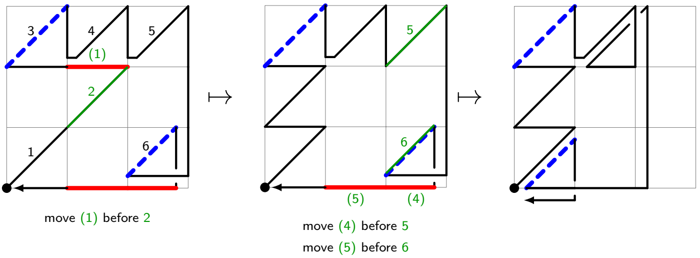

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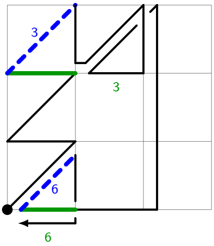

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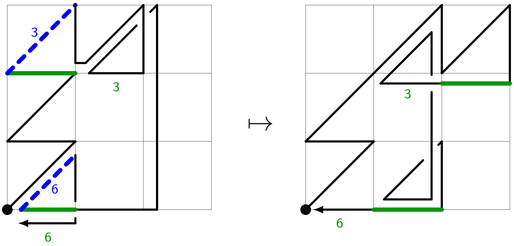

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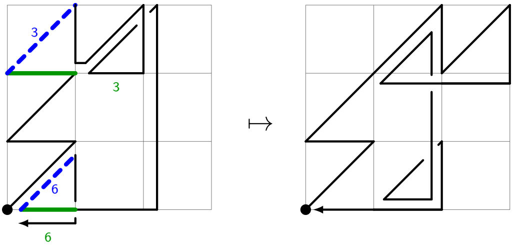

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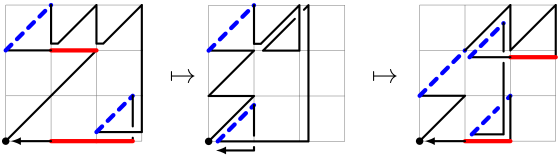
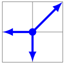
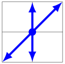
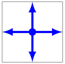
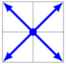


Kreweras excursions avoiding patterns of length 2

			
Kreweras	Gessel	Pólya	Diagonal

Pattern p	**#** $\{$**Kreweras excursions of length** $3n$ **avoiding** $p\}$	**OEIS**	**In bijection with**
	1, 2, 11, 85, 782, 8004, ...	A135404	Gessel excursions
	1, 2, 11, 85, 782, 8004, ...	A135404	Gessel excursions
	1, 1, 5, 37, 332, 3343, ...	None	Gessel excursions ending with $\downarrow$
	1, 2, 10, 70, 588, 5544, ...	A005568	Pólya excursions
	1, 1, 4, 25, 196, 1764, ...	A001246	Diagonal excursions

Coincidence?

Nature of generating functions? (Simplified)

Rational

$G(x) = \frac{p(x)}{q(x)}$ p, q polynomials

$F(x) = \frac{x}{1 - x - x^2}$ Fibonacci

Algebraic

P(x,y) non-zero polynomial: $P(x, G(x)) = 0$

$C(x) = 1 + xC(x)^2$ Catalan

$C(x) = \frac{1 - \sqrt{1 - 4x}}{2x}$

Differentiably finite (D-finite)

$a_i(x)$ not all zero polynomials

$\sum_{i=0}^{r} a_i(x) G^{(i)}(x) = 0$ r fixed integer

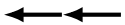
Harmonic numbers $H(x) = \frac{1}{1 - x} \log\left(\frac{1}{1 - x}\right)$

Transcendental

Not D-finite

$B(x) = \frac{x}{e^x - 1}$ Bernoulli numbers

Nature of generating functions in specific cases

Pattern p	**#** $\{$**Kreweras excursions of length** $3n$ **avoiding** $p\}$	**OEIS**	**Nature of gen. func.**
	1, 2, 11, 85, 782, 8004, ...	A135404	Algebraic
	1, 2, 11, 85, 782, 8004, ...	A135404	Algebraic
	1, 1, 5, 37, 332, 3343, ...	None	Algebraic
	1, 2, 10, 70, 588, 5544, ...	A005568	D-finite
	1, 1, 4, 25, 196, 1764, ...	A001246	D-finite

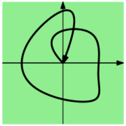
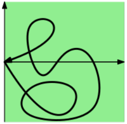
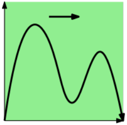
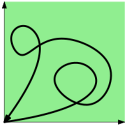
First three models: Algebraic

(using [Kauers–Koutschan–Zeilberger 2009] and [Bostan–Kauers 2009])
but not context-free [Banderier–Drmota 2015]: $4^n n^{-2/3}$ asymptotics.

Last two models: D-finite, but not algebraic.

Pólya walks: $C_n C_{n+1} \sim 4 \frac{16^n}{\pi n^3}$. Such asymptotics involving a n^{-3} factor are not compatible with the rather constrained asymptotics of algebraic function coefficients.

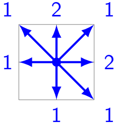
Generating functions: walks avoiding a pattern

For walks in $\mathbb{Z}^2$ avoiding a pattern, generating function is rational.  complement of regular expression	For walks in $\mathbb{N} \times \mathbb{Z}$ avoiding a pattern, generating function is algebraic.  pushdown automaton with single stack
For directed walks in $\mathbb{N}^2$ avoiding a pattern, generating function is algebraic.  vectorial kernel method [Asinowski, Bacher, Banderier, Gittenberger, Roitner]	For non-directed walks in $\mathbb{N}^2$ avoiding a pattern, gen. function is not necessarily algebraic.  Classification needed!

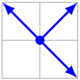
Generating functions: walks avoiding a pattern

For non-directed walks in $\mathbb{N}^2$ avoiding a pattern, the generating function is not necessarily algebraic (or D-finite!).

Kauers–Yatchak's model, 2015
(steps are with multiplicity)



Mishna–Rechnitzer's model, 2009



Forbidding some steps in the top (algebraic) model leads to the bottom (differentially transcendental) model.

Conclusion

Open Problem

Find a "Proof from The Book" for the Kreweras enumeration!

Perhaps by following some patterns...

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Even some small but reasonably general results in this direction... (small steps?)

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A highly symmetric walk?

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A highly symmetric walk?

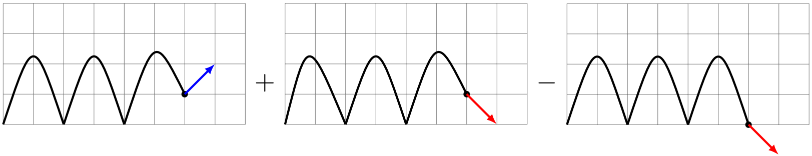
Thank you!

The kernel method: Dyck paths [Knuth 1968]

Let $F(z,u)$ be the generating function for Dyck paths with length marked by z and height marked by u, and $f_n(u)$ be defined as

$$F(z,u) = \sum_{n\geq 0} f_n(u) z^n.$$

The allowed steps are $(1,1)$ and $(1,-1)$, and so at each time step, the evolution of the height is encoded by a multiplication by the (Laurent) polynomial $P(u) = u^{-1} + u$.



$$f_{n+1}(u) = f_n(u)P(u) - \{u^{<0}\} f_n(u)P(u) = f_n(u)P(u) - u^{-1} f_n(0).$$

Summing over all n,

$$\sum_{n\geq 0} f_{n+1}(u)z^{n+1} = \sum_{n\geq 0} f_n(u)P(u)z^{n+1} - \sum_{n\geq 0} u^{-1}f_n(0)z^{n+1}$$

Simplifying,

$$F(z,u) - f_0(u) = P(u)zF(z,u) - zu^{-1}F(z,0)$$

⇝ functional equation:

$$(1 - zP(u))F(z,u) = 1 - z/uF(z,0)$$

The kernel $1 - zP(u)$ has two roots

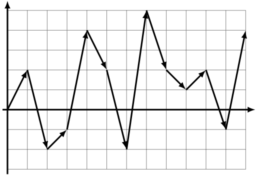
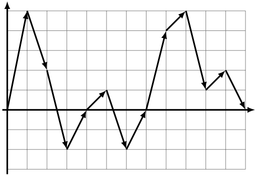
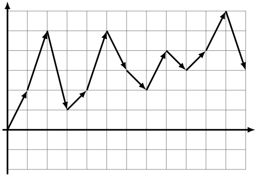
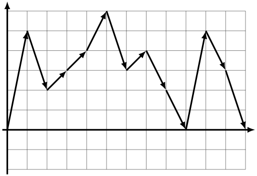
$$u_1(z) = \frac{1-\sqrt{1-4z^2}}{2z}, \qquad u_2(z) = \frac{1+\sqrt{1-4z^2}}{2z}.$$

$$u_1(z) \to 0 \text{ as } z \to 0, \qquad u_2(z) \to \infty \text{ as } z \to 0.$$

Plugging $u = u_1$, we get

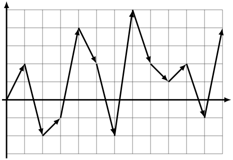
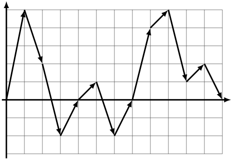
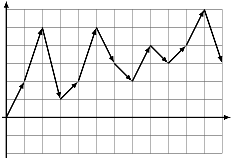
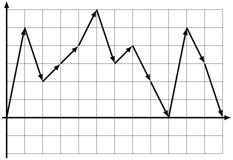
$$F(z,0) = \frac{u_1(z)}{z}$$

The kernel method: general paths [Banderier–Flajolet 2002]

	Ending anywhere	Ending at $y = 0$
Unconstrained	Walk  $W(z,u) = \frac{1}{1 - zP(u)}$	Bridge  $B(z) = z \sum_{i=1}^{c} \frac{u_i'(z)}{u_i(z)}$
Constrained $\mathbb{Z}_{\geq 0}$	Meander  $M(z,u) = \frac{u^{-c}}{1 - zP(u)} \prod_{i=1}^{c} (u - u_i(z))$	Excursion  $E(z) = \frac{(-1)^{c-1}}{z} \prod_{i=1}^{c} u_i(z)$

u_i = small roots (i.e. $u_i(z) \sim 0$ for $z \sim 0$) of the kernel $K(z,u) = 1 - zP(u)$

Enumeration of directed lattice paths (marking m patterns) [Asinowski–Bacher–Banderier–Gittenberger–Roitner]

	Ending anywhere	Ending at $y=0$
Unconstrained	Walk  $W(z,u,v_1,\dots,v_m) = \frac{\Delta(z,u)}{K(z,u)}$	Bridge  $B(z,v_1,\dots,v_m) = -\sum_{i=1}^{e} \frac{u_i'}{u_i} \frac{\Delta(z,u_i)}{\partial_z K(z,u_i)}$
Constrained $\mathbb{Z}_{\geq 0}$	Meander  $M(z,u,v_1,\dots,v_m) = \frac{\Delta(z,u)}{u^e K(z,u)} \prod_{i=1}^{e}(u - u_i(z))$	Excursion  $E(z,v_1,\dots,v_m) = \frac{(-1)^{e+1}}{z} \prod_{i=1}^{e} u_i(z)$

u_i= small roots of the kernel $K(z,u) = \det(I - zA)$ "$= 1 - zP(u)$".
$\Delta(z,u)$ is the determinant of the mutual correlation matrix.