Jack characters as generating series of bipartite maps and proof of Lassalle's conjecture

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joint work with

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Symmetric functions

We consider the space of symmetric functions on an alphabet $\mathbf{x} := x_1, x_2, \ldots$.

For $k \geq 0$, the power sum function p_k is defined

$$p_k(\mathbf{x}) := \sum_{i \geq 1} x_i^k,$$

and if $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_\ell]$ then

$$p_\lambda(\mathbf{x}) = p_{\lambda_1}(\mathbf{x}) p_{\lambda_2}(\mathbf{x}) \cdots p_{\lambda_\ell}(\mathbf{x}).$$

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The expansion of Schur functions on the power-sum basis is given by

$$s_\lambda(\mathbf{x}) = \sum_{\mu \vdash n} \frac{\chi^\lambda(\mu)}{z_\mu} p_\mu(\mathbf{x}) \qquad \text{where } z_\lambda := \frac{|\lambda|!}{|\mathcal{C}_\lambda|}.$$

Jack polynomials

Jack polynomials $J_\lambda^{(\alpha)}$ are symmetric functions which depend on a deformation parameter α.

- They can be obtained from Macdonald polynomials $J_\lambda^{(q,t)}$ by taking $q = t^\alpha$ and the limit $t \to 1$.
- When we take $\alpha = 1$ we obtain Schur functions.

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- generalizes a known interpretation of Schur functions in terms of pairs of permutations/oriented bipartite maps.
- answers a positivity conjecture of Lassalle 2008.
- gives an answer in some sense to a conjecture of Hanlon 1988.

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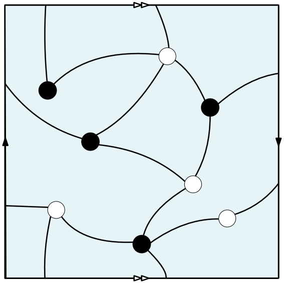
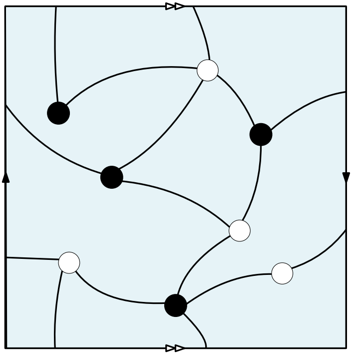
- A map is *oriented* if its all the connected components are embedded into orientable surfaces.
- A map is bipartite if its vertices are colored in white and black, and each white vertex has only black neighbors.



A non-oriented bipartite map on the Klein bottle.

Maps

- The face-type of a bipartite map M, denoted by $\diamond(M)$, is the is the partition given by the face degrees, divided by 2.

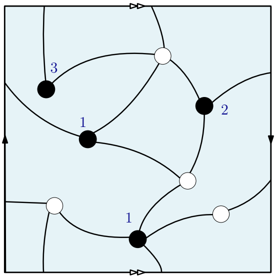


A non-oriented map of face-type $[4, 4, 2, 2]$.

Layered maps

Let k be a positive integer. A map M is k-layered if

- each black vertex has a label in $1, 2, \dots, k$.



A 3-layered map on the Klein bottle

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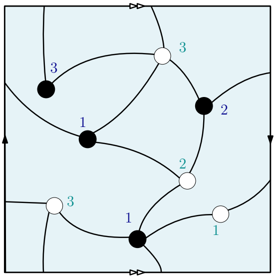

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Definition (Goulden–Jackson '96)

A statistic of non-orientability (on k-layered maps) is a statistic which associates to each k-layered map M a non-negative integer such that $\vartheta(M) = 0$ if and only if M is oriented.

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Maps will be counted with a weight $b^{\vartheta(M)}$, where $b := \alpha - 1$ is the shifted Jack parameter.

Jack polynomials in the power-sum basis

Theorem (BD–Dołęga '23)

There exists an explicit statistic of non-orientability ϑ, such that

$$J_\lambda^{(\alpha)} = (-1)^{|\lambda|} \sum_{\substack{\ell(\lambda)-layered \\ maps\ M}} \frac{p_{\diamond(M)} b^{\vartheta(M)}}{2^{|\mathcal{V}_\bullet(M)|-cc(M)} \alpha^{cc(M)}} \prod_{1\leq i\leq \ell(\lambda)} \frac{(-\alpha\lambda_i)^{|\mathcal{V}_\circ^{(i)}(M)|}}{z_{\nu_\bullet^{(i)}(M)}},$$

- $p_{\diamond(M)}$ *is the power-sum function associated to the partition* $\diamond(M)$
- $|\mathcal{V}_\bullet(M)|$ *is the number of black vertices of* M.
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- a face-weight $p_{\diamond(M)}$
- a non-orientability weight $b^{\vartheta(M)}$
- a weight related to layers structure $(-\alpha\lambda_i)^{|\mathcal{V}_\circ^{(i)}(M)|}$

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Well known for $\alpha = 1$ (Young symmetrizers) and for $\alpha = 2$ (Féray–Śniady's 2010).

Jack characters (a dual approach)

Fix a partition μ.

$$\theta_\mu^{(\alpha)}(\lambda) := \begin{cases} 0, & \text{if } |\lambda| < |\mu|. \\ \binom{|\lambda|-|\mu|+m_1(\mu)}{m_1(\mu)} [p_{\mu,1^{|\lambda|-|\mu|}}] J_\lambda^{(\alpha)}, & \text{if } |\lambda| \geq |\mu|. \end{cases}$$

where $m_1(\mu)$ is the number of parts of size 1 in μ.

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There exists a statistic of non-orientability ϑ on layered maps, such that

$$\theta_\mu^{(\alpha)}(\lambda) = (-1)^{|\mu|} \sum_{\substack{\text{layered maps} M \\ \text{of face-type } \mu}} \frac{b^{\vartheta(M)}}{2^{|\mathcal{V}_\bullet(M)|-cc(M)} \alpha^{cc(M)}} \prod_{i \geq 1} \frac{(-\alpha\lambda_i)^{|\mathcal{V}_\circ^{(i)}(M)|}}{z_{\nu_\bullet^{(i)}(M)}}, \quad (1)$$

- For $\alpha = 1$: Stanley-Féray formula 2010.
- For $\alpha = 2$: Féray–Śniady formula for zonal characters 2010.

Idea of the proof

Known: There exists a unique α-shifted symmetric function $f_\mu(u_1, u_2, \dots)$ (i.e symmetric in the variables $u_1 - 1/\alpha, u_2 - 2/\alpha, \dots$) such that

$$\theta_\mu^{(\alpha)}(\lambda) = f_\mu(\lambda_1, \lambda_2, \dots, \lambda_\ell, 0, \dots) \text{ for every } \lambda.$$

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Fix a partition μ. The Jack character $\theta_\mu^{(\alpha)}$ is the unique α-shifted symmetric function of degree $|\mu|$ with top homogeneous part $\alpha^{|\mu|-\ell(\mu)}/z_\mu \cdot p_\mu$, such that $\theta_\mu^{(\alpha)}(\lambda) = 0$ for any partition $|\lambda| < |\mu|$.

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- We introduce the generating series of k-layered maps

$$F^{(k)}(t, \mathbf{p}, s_1, \dots, s_k) :=$$

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We prove that this generating series satisfies the three conditions of the chracterization theorem.

Idea of the proof

- For a well-chosen statistic of non-orientability ϑ, this generating series can be constructed inductively using differential operators (Tutte decomposition):

$$F^{(k)}(t, \mathbf{p}, s_1, \dots, s_k) = \exp\left(\sum_{n\geq 1} \frac{(-t)^n}{n} B_n(\mathbf{p}, -\alpha s_1)\right) \cdot F^{(k-1)}(t, \mathbf{p}, s_2, \dots, s_k),$$

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where

$$B_n(\mathbf{p}, -\alpha s_1) := \Theta_Y \left(\Gamma_Y - \alpha s_1 Y_+\right)^n \frac{y_0}{1+b}$$

is an operator which adds a black vertex of degree n with label 1.
$Y := (y_0, y_1, y_2, \dots,)$ is a catalytic variable, and

$$\left.\begin{aligned} \Theta_Y &:= \textstyle\sum_{i\geq 1} p_i \frac{\partial}{\partial y_i}, \qquad Y_+ = \sum_{i\geq 0} y_{i+1} \frac{\partial}{\partial y_i}, \\ \Gamma_Y &= (1+b) \cdot \textstyle\sum_{i,j\geq 1} y_{i+j} \frac{i\partial^2}{\partial p_i \partial y_{j-1}} \\ &\quad + \textstyle\sum_{i,j\geq 1} y_i p_j \frac{\partial}{\partial y_{i+j-1}} + b \cdot \sum_{i\geq 1} y_{i+1} \frac{i\partial}{\partial y_i}. \end{aligned}\right\} \text{Chapuy–Dołęga operators.}$$

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A key step of the proof: Two commutation relations

$$[B_n(\mathbf{p}, u), B_m(\mathbf{p}, u)] = 0, \text{ for } n, m \geq 1,$$

$$\left[\sum_{n\geq 1} \frac{t^n}{n} B_n^{>}(\mathbf{p}, u), \sum_{n\geq 1} \frac{t^n}{n} B_n^{>}(\mathbf{p}, v)\right] = 0,$$

where

$$B_n^{>}(\mathbf{p}, u) := B_n(\mathbf{p}, u) - B_n(\mathbf{p}, 0).$$

Application 1: Creation operators for Jack polynomials

Theorem

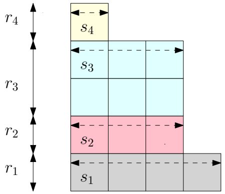
$$J^{(\alpha)}_{(\lambda_1,\lambda_2,\dots,\lambda_\ell)} = \mathcal{B}^{(+)}_{\lambda_1}\cdot\mathcal{B}^{(+)}_{\lambda_2}\cdots\mathcal{B}^{(+)}_{\lambda_\ell}\cdot 1,$$

where

$$\mathcal{B}^{(+)}_{n} := [t^n]\exp\left(\sum_{n\geq 1}\frac{(-t)^n}{n}B_n(\mathbf{p},-\alpha n)\right)$$

Application 2: Lassalle's conjecture 2008

Stanley's coordinates of a Young diagram



The Young diagram of the partition $[4, 3, 3, 3, 1]$, with $\mathbf{s} = (4, 3, 3, 1)$ and $\mathbf{r} = (1, 1, 2, 1)$ as Stanley coordinates.

Theorem (Lassalle's conjecture on Jack characters)

The normalized Jack character $(-1)^{|\mu|} z_\mu \theta_\mu^{(\alpha)}$ *is a polynomial in Stanley's coordinates* $r_1, r_2 \ldots, -s_1, -s_2 \ldots,$ *and* b *with non-negative integer coefficients.*

Proof:

- Non-negativity: the combinatorial interpretation in terms of maps.
- Integrality: (a different approach) the integrable system of Nazarov–Sklyanin.