Jack characters as generating series of bipartite maps and proof of Lassalle’s conjecture

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Symmetric functions

We consider the space of symmetric functions on an alphabet $\mathbf{x} := x_1, x_2, \ldots$.

For $k \geq 0$, the power sum function $p_k$ is defined

$$ p_k(\mathbf{x}) := \sum_{i \geq 1} x_i^k, $$

and if $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_\ell]$ then

$$ p_\lambda(\mathbf{x}) = p_{\lambda_1}(\mathbf{x})p_{\lambda_2}(\mathbf{x}) \cdots p_{\lambda_\ell}(\mathbf{x}). $$
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\]

The expansion of Schur functions on the power-sum basis is given by

\[
s_\lambda(x) = \sum_{\mu \vdash n} \frac{\chi_\lambda(\mu)}{z_\mu} p_\mu(x) \quad \text{where} \quad z_\lambda := \frac{|\lambda|!}{|C_\lambda|}.
\]
Jack polynomials

Jack polynomials $J^{(\alpha)}_\lambda$ are symmetric functions which depend on a deformation parameter $\alpha$.

- They can be obtained from Macdonald polynomials $J^{(q,t)}_\lambda$ by taking $q = t^\alpha$ and the limit $t \to 1$.
- When we take $\alpha = 1$ we obtain Schur functions.
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Main result

A combinatorial interpretation of the power-sum expansion of Jack polynomials in terms of bipartite maps (oriented or not).
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- answers a positivity conjecture of Lassalle 2008.
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- generalizes a known interpretation of Schur functions in terms of pairs of permutations/oriented bipartite maps.
- answers a positivity conjecture of Lassalle 2008.
- gives an answer in some sense to a conjecture of Hanlon 1988.
Maps

- A *connected map* is a connected graph embedded into a surface, **oriented or not**. A map is a collection of connected maps.
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- A map is *oriented* if its all the connected components are embedded into orientable surfaces.
- A map is bipartite if its vertices are colored in white and black, and each white vertex has only black neighbors.

A non-oriented bipartite map on the Klein bottle.
Maps

- The **face-type** of a bipartite map \( M \), denoted by \( \diamond(M) \), is the partition given by the face degrees, divided by 2.

A non-oriented map of face-type \([4, 4, 2, 2]\).
Layered maps

Let $k$ be a positive integer. A map $M$ is $k$-layered if

- each black vertex has a label in $1, 2, \ldots, k$.

A 3-layered map on the Klein bottle
Layered maps

Let $k$ be a positive integer. A map $M$ is $k$-layered if

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**Definition (Goulden–Jackson ’96)**

A statistic of non-orientability (on $k$-layered maps) is a statistic which associates to each $k$-layered map $M$ a non-negative integer such that $\vartheta(M) = 0$ if and only if $M$ is oriented.
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Maps will be counted with a weight $b^{\vartheta(M)}$, where $b := \alpha - 1$ is the shifted Jack parameter.
Theorem (BD–Dołęga ’23)

There exists an explicit statistic of non-orientability $\vartheta$, such that

$$
J^{(\alpha)}_{\lambda} = (-1)^{|\lambda|} \sum_{\ell(\lambda)\text{-layered maps } M} \frac{p_{\circ}(M) b^{\vartheta(M)}}{2|\mathcal{V}_{\bullet}(M)| - cc(M) \alpha cc(M)} \prod_{1 \leq i \leq \ell(\lambda)} \frac{(-\alpha \lambda_i)|\nu_{\circ}^{(i)}(M)|}{z_{\nu_{\circ}^{(i)}}(M)},
$$

- $p_{\circ}(M)$ is the power-sum function associated to the partition $\circ(M)$
- $|\mathcal{V}_{\bullet}(M)|$ is the number of black vertices of $M$.
- $|\nu_{\circ}^{(i)}(M)|$ is the number of white vertices of $M$ labelled by $i$.
- $cc(M)$ is the number of connected components of $M$. 
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$$J_{\lambda}^{(\alpha)} = (-1)^{|\lambda|} \sum_{\ell(\lambda)-layered \ maps \ M} \frac{p_{\diamond}(M)b^{\vartheta(M)}}{2|V_\bullet(M)| - cc(M)\alpha cc(M)} \prod_{1 \leq i \leq \ell(\lambda)} \frac{(-\alpha \lambda_i)|\nu^{(i)}_\circ(M)|}{z^{(i)}_{\nu_\bullet}(M)},$$

- $p_{\diamond}(M)$ is the power-sum function associated to the partition $\diamond(M)$
- $|V_\bullet(M)|$ is the number of black vertices of $M$.
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- a face-weight $p_{\diamond}(M)$
- a non-orientability weight $b^{\vartheta(M)}$
- a weight related to layers structure $(-\alpha \lambda_i)|\nu^{(i)}_\circ(M)|$
Jack polynomials in the power-sum basis

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- \( cc(M) \) is the number of connected components of \( M \).

Well known for \( \alpha = 1 \) (Young symmetrizers) and for \( \alpha = 2 \) (Féray–Śniady’s 2010).
Jack characters (a dual approach)

Fix a partition $\mu$.

$$\theta^{(\alpha)}_{\mu}(\lambda) := \begin{cases} 0, & \text{if } |\lambda| < |\mu|. \\ (|\lambda| - |\mu| + m_1(\mu)) \left[p_{\mu,1|\lambda|-|\mu|} \right] J^{(\alpha)}_{\lambda}, & \text{if } |\lambda| \geq |\mu|. \end{cases}$$

where $m_1(\mu)$ is the number of parts of size 1 in $\mu$. 

Theorem (BD–Dołęga '23)

There exists a statistic of non-orientability $\vartheta$ on layered maps, such that

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For $\alpha = 1$: Stanley-Féray formula 2010.

For $\alpha = 2$: Féray–Śniady formula for zonal characters 2010.
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There exists a statistic of non-orientability $\vartheta$ on layered maps, such that

$$\theta^{(\alpha)}_{\mu}(\lambda) = (-1)^{|\mu|} \sum_{\substack{\text{layered maps } M \\ \text{of face-type } \mu}} \frac{b^{\vartheta(M)}}{2|V_\bullet(M)| - cc(M) \alpha^{cc(M)}} \prod_{i \geq 1} \frac{(-\alpha \lambda_i)|V^{(i)}_{\nu}(M)|}{Z_{\nu}^{(i)}(M)}, \quad (1)$$

- For $\alpha = 1$: Stanley-Féray formula 2010.
- For $\alpha = 2$: Féray–Śniady formula for zonal characters 2010.
Idea of the proof

**Known:** There exists a unique $\alpha$-shifted symmetric function $f_\mu(u_1, u_2, \ldots)$ (i.e symmetric in the variables $u_1 - 1/\alpha, u_2 - 2/\alpha, \ldots$) such that

$$\theta^{(\alpha)}(\lambda) = f_\mu(\lambda_1, \lambda_2, \ldots, \lambda_\ell, 0, \ldots)$$

for every $\lambda$. 
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**Theorem (Féray ’19)**

Fix a partition $\mu$. The Jack character $\theta_\mu^{(\alpha)}$ is the unique $\alpha$-shifted symmetric function of degree $|\mu|$ with top homogeneous part $\alpha^{|\mu| - \ell(\mu)}/z_\mu \cdot p_\mu$, such that

$$\theta_\mu^{(\alpha)}(\lambda) = 0$$

for any partition $|\lambda| < |\mu|$. 
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- We introduce the generating series of $k$-layered maps

$$F^{(k)}(t, p, s_1, \ldots, s_k) :=$$

$$\sum_{k\text{-layered maps } M} (-t)^{\varnothing(M)} p^{\varnothing(M)} \frac{b^{\varnothing(M)}}{2^{|\nu\cdot(M)|-cc(M)} \alpha^{cc(M)}} \prod_{1 \leq i \leq k} \frac{(-\alpha s_i)^{|\nu^{(i)}(M)|}}{z_{\nu^{(i)}(M)}}.$$
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**Theorem (Féray ’19)**

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- We introduce the generating series of $k$-layered maps

$$F^{(k)}(t, p, s_1, \ldots, s_k) := \sum_{k\text{-layered maps } M} (-t)^{\diamond(M)} p^{\diamond(M)} \frac{b^{\theta(M)}}{2|\mathcal{V}(M)| - cc(M) \alpha cc(M)} \prod_{1 \leq i \leq k} \frac{(-\alpha s_i)^{|\mathcal{V}_i(M)|}}{z_{\mathcal{V}_i(M)}}.$$

We prove that this generating series satisfies the three conditions of the characterization theorem.
Idea of the proof

- For a well-chosen statistic of non-orientability $\vartheta$, this generating series can be constructed inductively using differential operators (Tutte decomposition):

$$F^{(k)}(t, p, s_1, \ldots, s_k) = \exp \left( \sum_{n \geq 1} \frac{(-t)^n}{n} B_n(p, -\alpha s_1) \right) \cdot F^{(k-1)}(t, p, s_2, \ldots, s_k),$$
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\]

where

\[
B_n(p, -\alpha s_1) := \Theta_Y (\Gamma_Y - \alpha s_1 Y_+)^n \frac{y_0}{1 + b}
\]

is an operator which adds a black vertex of degree \( n \) with label 1. \( Y := (y_0, y_1, y_2, \ldots, ) \) is a catalytic variable, and

\[
\Theta_Y := \sum_{i \geq 1} p_i \frac{\partial}{\partial y_i}, \quad Y_+ = \sum_{i \geq 0} y_{i+1} \frac{\partial}{\partial y_i},
\]

\[
\Gamma_Y = (1 + b) \cdot \sum_{i,j \geq 1} y_{i+j} \frac{i \partial^2}{\partial p_i \partial y_{j-1}} + \sum_{i,j \geq 1} y_i p_j \frac{\partial}{\partial y_{i+j-1}} + b \cdot \sum_{i \geq 1} y_{i+1} \frac{i \partial}{\partial y_i}.
\]

Chapuy–Dołęga operators.
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A key step of the proof: Two commutation relations

$$[B_n(p, u), B_m(p, u)] = 0, \text{ for } n, m \geq 1,$$

$$\left[ \sum_{n \geq 1} \frac{t^n}{n} B_n^>(p, u), \sum_{n \geq 1} \frac{t^n}{n} B_n^>(p, v) \right] = 0,$$

where

$$B_n^>(p, u) := B_n(p, u) - B_n(p, 0).$$
Application 1: Creation operators for Jack polynomials

Theorem

\[ J^{(\alpha)}_{(\lambda_1, \lambda_2, \ldots, \lambda_\ell)} = B^{(+)}_{\lambda_1} \cdot B^{(+)}_{\lambda_2} \cdots B^{(+)}_{\lambda_\ell} \cdot 1, \]

where

\[ B^{(+)}_n := [t^n] \exp \left( \sum_{n \geq 1} \frac{(-t)^n}{n} B_n(p, -\alpha n) \right) \]
The Young diagram of the partition $[4,3,3,3,1]$, with $s = (4,3,3,1)$ and $r = (1,1,2,1)$ as Stanley coordinates.

**Theorem (Lassalle’s conjecture on Jack characters)**

The normalized Jack character $(-1)^{|\mu|} z_\mu \theta^{(\alpha)}_\mu$ is a polynomial in Stanley’s coordinates $r_1, r_2, \ldots, -s_1, -s_2, \ldots$, and $b$ with non-negative integer coefficients.

**Proof:**

- Non-negativity: the combinatorial interpretation in terms of maps.
- Integrality: (a different approach) the integrable system of Nazarov–Sklyanin.