The multispecies zero range process and modified Macdonald polynomials

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Macdonald polynomials and hopping particles

- Macdonald polynomials (Macdonald '88) $P_\lambda(X; q, t)$ are a remarkable family of symmetric functions with coefficients in $\mathbb{Q}[q, t]$ that simultaneously generalize Schur functions ($q = t$), Hall-Littlewood polynomials ($q = 0$), Jack polynomials ($t = q^\alpha$ and $q \to 1$), others..

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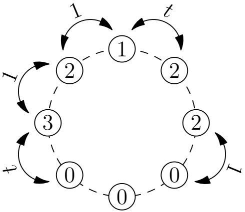
- the P_λ's are connected to a 1D particle model called the multispecies asymmetric simple exclusion process (ASEP) with parameter $0 \leq t \leq 1$ and particle types given by λ



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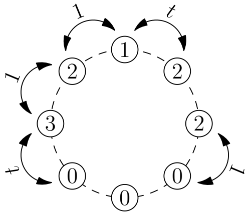

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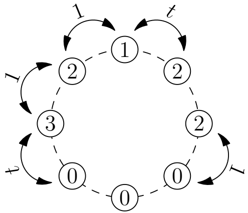

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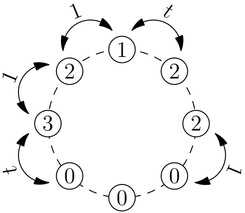

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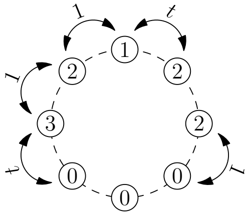

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- $P_\lambda(x_1, \ldots, x_n; q, t)$ specializes to the partition function of the multispecies ASEP at $x_1 = \cdots = x_n = q = 1$ (Cantini–de Gier–Wheeler '15)

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the stationary distribution of the ASEP can be described using multiline queues

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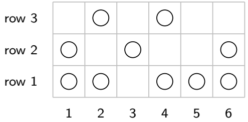

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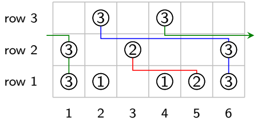

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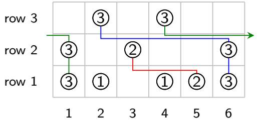

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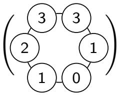

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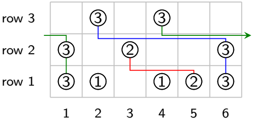

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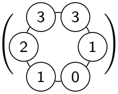

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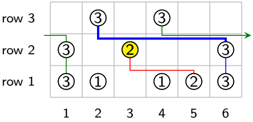

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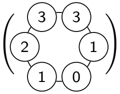
$\in$ MLQ 

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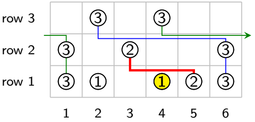

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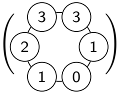
$\in$ MLQ 

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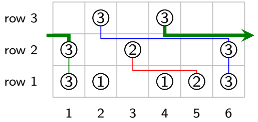

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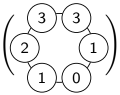

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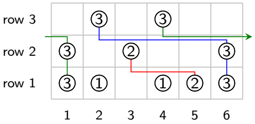

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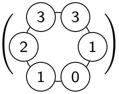

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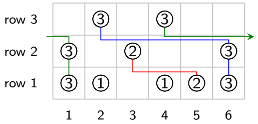

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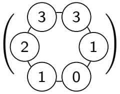

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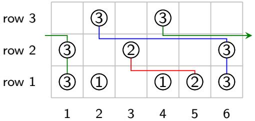

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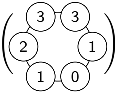

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- Can be represented by a tableau, where each string is mapped to a column

multiline queues, the ASEP, and Macdonald polynomials

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Theorem (Cantini-deGier-Wheeler '15)

At $x_1 = \cdots = x_n = q = 1$, P_λ specializes to the partition function of ASEP(λ, n)*:*

$$P_\lambda(1,\ldots,1;1,t) = \sum_{\tau \in S_n \cdot \lambda} \widetilde{\Pr}(\tau)$$

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Fix λ, n. The (unnormalized) stationary probability of $\tau \in \mathrm{ASEP}(\lambda, n)$ is

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Theorem (Corteel-M-Williams '18)

The Macdonald polynomial is given by

$$P_\lambda(x_1,\ldots,x_n;q,t) = \sum_{M \in \mathrm{MLQ}(\lambda,n)} \mathrm{wt}(M)(X;q,t)$$

modified Macdonald polynomials

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(J_λ is a scalar multiple of P_λ)

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$$\widetilde{H}_{(2,1)} = qt\ s_{(1,1,1)} + (q + t)s_{(2,1)} + s_3$$

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Our goal is to get a multiline queue-esque construction for $\widetilde{H}_\lambda(X; q, t)$ by interpreting plethysm through multiline queues

From multiline queues to $\widetilde{H}_\lambda$

$$\widetilde{H}_\lambda(X; q, t) = f_\lambda(q,t)\ P_\lambda\left[\frac{X}{1-t^{-1}}; q, t^{-1}\right]$$
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this can be represented by a multiline queue with columns labeled $x_1, x_1 t^{-1}, x_1 t^{-2}, \ldots, x_2, x_2 t^{-1}, x_2 t^{-2}, \ldots$

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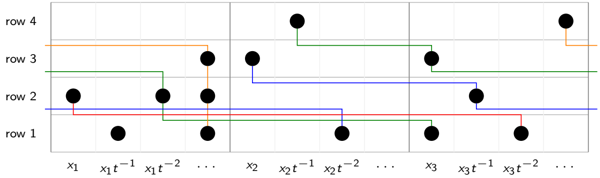

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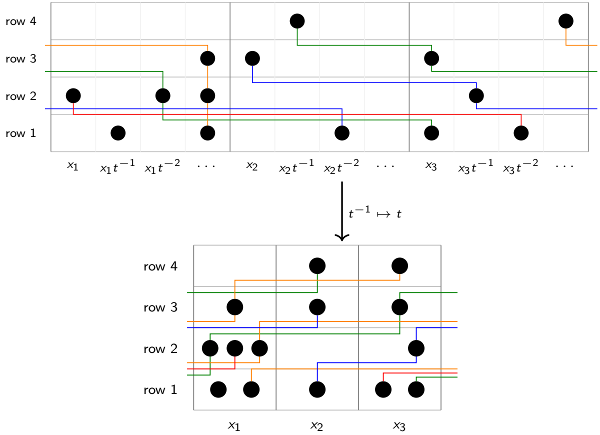

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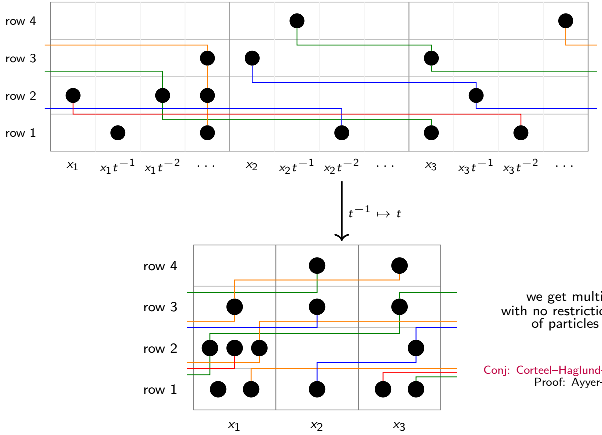
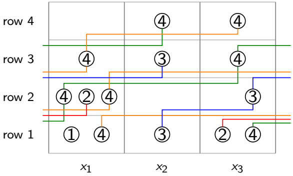


we get multiline diagrams
with no restriction on the number
of particles at each site!

Conj: Corteel–Haglund–M–Mason–Williams '20
Proof: Ayyer–M–Martin '21

From multiline diagrams to tableaux

Each **string** is mapped to a **column** in the tableau:



$\longrightarrow$ $\sigma =$

2	3			
3	1	2		
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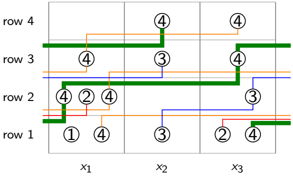

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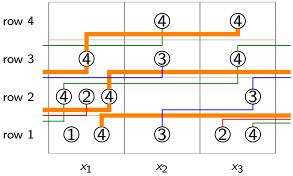

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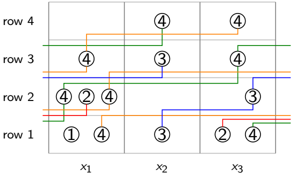

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column content of the diagram $\rightarrow$ the monomial weight x^σ

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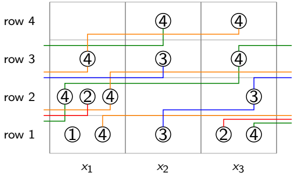

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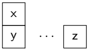

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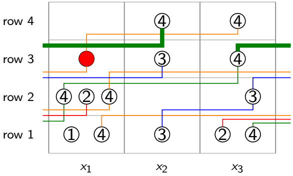

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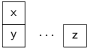

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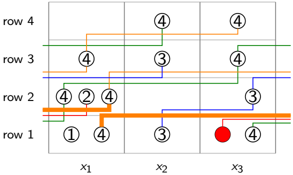

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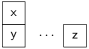

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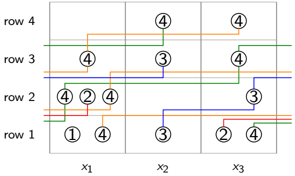

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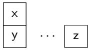

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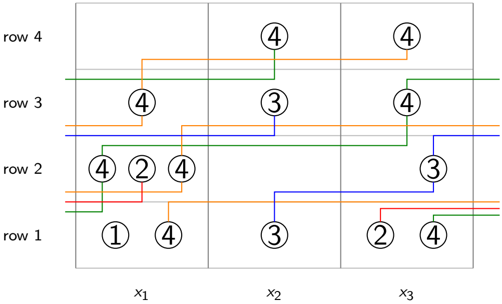
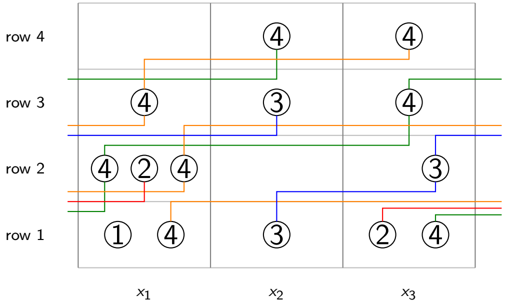
Theorem (Ayyer–M–Martin '21)

Let λ be a partition. The modified Macdonald polynomial equals

$$\widetilde{H}_\lambda(x_1, \ldots, x_n; q, t) = \sum_{\sigma:\lambda' \to [n]} q^{\text{maj}(\sigma)} t^{\text{quinv}(\sigma)} x^\sigma$$



The particle process corresponding to multiline diagrams is the multispecies totally asymmetric zero range process!



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The state of the particle process is read off the bottom row of the diagram:

$$\tau = (4, 1 \mid 3 \mid 4, 1)$$

multispecies totally asymmetric zero range process

- a zero range process (ZRP) is continuous-time 1D stochastic process (Spitzer '70). Each site can contain any number of particles, and particles hop from site j to site $j \pm 1$ with rates that depend only on the content of site j.

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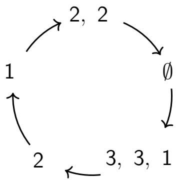
- In our case, we have a circular lattice with n sites, particles of types $\{\lambda_1, \lambda_2, \ldots\}$, which are moving clockwise.



Here, $n = 5$, $\lambda = (3, 3, 2, 2, 2, 1, 1)$

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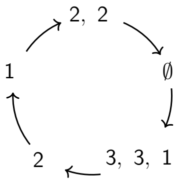

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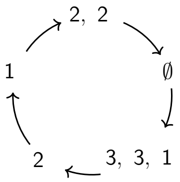

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$$x_j^{-1} t^m,$$

where m is the number of particles at site j that have **larger type** than ℓ.

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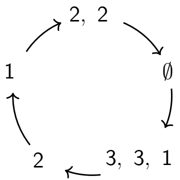

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where m is the number of particles at site j that have **larger type** than ℓ.
- Kuniba–Maruyama–Okado (2015+) (and others) have studied many variants of the TAZRP (all of which are integrable!). The version we study was first studied by Takayama '15

TAZRP probabilities and tableaux

Theorem (Ayyer–M–Martin '21)

Fix λ, n. The (unnormalized) stationary probability of $\tau \in \mathrm{TAZRP}(\lambda, n)$ *is*

$$\widetilde{\Pr}(\tau) = \sum_{\substack{\sigma:\lambda' \to [n] \\ \sigma \text{ has type } \tau}} x^{\sigma} t^{\mathrm{quinv}(\sigma)}.$$

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Corollary

The *partition function* *of* $\mathsf{TAZRP}(\lambda, n)$ *is*

$$\mathcal{Z}_{\lambda,n}(x_1, \ldots, x_n; t) = \widetilde{H}_{\lambda}(x_1, \ldots, x_n; 1, t).$$

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Proof: construction of a Markov chain on tableaux that lumps to the TAZRP.

a Markov chain on tableaux: transitions

- Each cell in the tableau is equipped with an exponential clock.

5			
2	1		
1	4		
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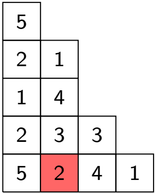

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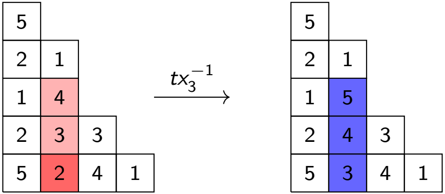

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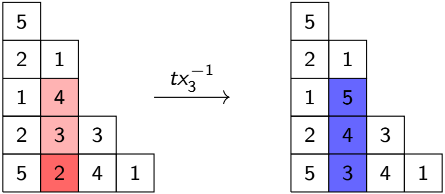

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The stationary distribution of the Markov process on tableaux $\sigma : \lambda' \to [n]$ is

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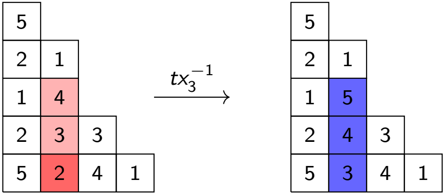

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- if $c = (1, j)$ is in the bottom row, the rate $f(\sigma, c)$ matches the transition rate $f_{\sigma(c)}(\lambda_j)$ of the corresponding particle in the TAZRP.
- (when λ has repeated parts, we need to do some more work!)

Observables

Observables are macroscopic properties of the TAZRP process that can be measured e.g. through simulation. These include:

- stationary probabilities/the partition function
- the current
- densities of particle content at individual sites
- correlations of behaviors of tuples of particles or tuples of sites

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what can we learn about the observables using the enhanced Markov chain on multiline diagrams/tableaux and the connection with modified Macdonald polynomials?

Current

The current of particle ℓ across the edge j is defined as the number of particles of type ℓ traversing the edge j per unit of time in the large time limit.

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Proposition (Current for the single species TAZRP)

For the single-species TAZRP on n sites with m particles, the current is given by

$$J = [m]_t \ \frac{\widetilde{H}_{(1^{m-1})}(x_1, \ldots, x_n; 1, t)}{\widetilde{H}_{(1^m)}(x_1, \ldots, x_n; 1, t)}.$$

Current

The current of particle ℓ across the edge j is defined as the number of particles of type ℓ traversing the edge j per unit of time in the large time limit.

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For the single-species TAZRP on n sites with m particles, the current is given by

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This comes from the stationary probability of the 1-species TAZRP:

$$\pi(\tau) = \frac{1}{\widetilde{H}_{(1^m)}(x_1, \ldots, x_n; 1, t)} \begin{bmatrix} m \\ \tau_1, \ldots, \tau_n \end{bmatrix}_t \prod_{i=1}^{n} x_i^{\tau_i}$$

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Theorem (Ayyer-M-Martin '22+)

Let $\lambda = (1^{m_1}, \ldots, k^{m_k})$, and let $1 \le j \le k$. The current of the particle of type j of the TAZRP of type λ on n sites is given by

$$J = \left[m_j + \cdots + m_k\right]_t \; \frac{\widetilde{H}_{\left(1^{m_j + \cdots + m_k - 1}\right)}}{\widetilde{H}_{\left(1^{m_j + \cdots + m_k}\right)}} - \left[m_{j+1} + \cdots + m_k\right]_t \; \frac{\widetilde{H}_{\left(1^{m_{j+1} + \cdots + m_k - 1}\right)}}{\widetilde{H}_{\left(1^{m_{j+1} + \cdots + m_k}\right)}}$$

Particle densities

- Take TAZRP(λ, n) with content $\lambda = (1^{m_1}, 2^{m_2}, \ldots, k^{m_k})$
- $z_j^{(\ell)}$: random variable counting the number of particles of type ℓ at site j in a configuration of TAZRP(λ, n)
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For $1 \leq \ell \leq k$, the density of the ℓ'th species at site 1 is given by

$$\langle z_1^{(\ell)} \rangle = x_1 \partial_{x_1} \log \left(\frac{\widetilde{H}_{(1^{m_\ell + \cdots + m_k})}(x_1, \dots, x_n; 1, t)}{\widetilde{H}_{(1^{m_{\ell+1} + \cdots + m_k})}(x_1, \dots, x_n; 1, t)} \right).$$

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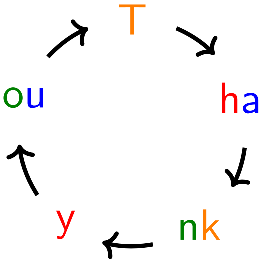
This generalizes to probabilities for fixed content on an interval of sites $1, \dots, k$: these probabilities are symmetric in the variables $\{x_{k+1}, \dots, x_n\}$!

Future directions

- a few other particle models have been found to be connected to Macdonald polynomials:
  - inhomogeneous ASEP and Schubert polynomials Lam–Williams '11
  - long-range ASEP and $P_\lambda(x_1, \ldots, x_n; 1, t)$ Angel–Ayyer–Martin '23+
  - multi-hopping multispecies TAZRP and $\widetilde{H}_\lambda(x_1, \ldots, x_n; 1, 0)$, $\widetilde{H}_\lambda(1, \ldots, 1; 1, t)$ Esipova–M '23+, Corteel–Keating '23+

  What other particle processes can be described through Macdonald (or related) polynomials?

- Using multiline queues, Corteel-Haglund-M-Mason-Williams FPSAC '20 defined quasisymmetric Macdonald polynomials which refine the symmetric Macdonald polynomial $P_\lambda(X; q, t)$. Can we use a parallel construction to define an interesting family of quasisymmetric polynomials that refine $\widetilde{H}_\lambda(X; q, t)$?



Modified Macdonald polynomials and the multispecies zero range process:
arXiv:2011.06117, arXiv:2209.09859

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- Example: let $\lambda = (2,2,1,1)$, $n = 4$, $\ell = 2$, and $w = (2|1)$. Configurations contributing to $\mathbb{P}_{\lambda,n}(\overline{w})$ are

$$(2|1|12|\cdot), \quad (2|1|1|2), \quad (2|1|2|1), \quad (2|1|\cdot|12)$$

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$\mathbb{P}_{\lambda,n}(\overline{w})$ is symmetric in the variables $\{x_{\ell+1}, \ldots, x_n\}$.