Divisors, orbit harmonics, and DT invariants

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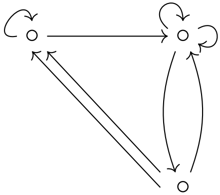
UCSD

Outline

- Cohomological Hall algebra $\mathcal{H}$
- Donaldson-Thomas invariants
- DT invariants via polytopes

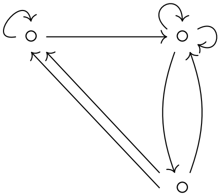
Quivers

Def: A *quiver* $Q = (Q_0, Q_1)$ is a directed graph. Loops and multiple edges are allowed.



Physics

Kontsevich-Soibelman: Given a quiver Q, define the *cohomological Hall algebra* $\mathcal{H}$.

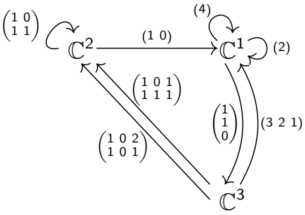


“ There is an old proposal in String Theory . . . which says that with a certain class of 4-dimensional quantum theories with $N = 2$ spacetime supersymmetry one should be able to associate an algebra graded by the charge lattice, called the algebra of BPS states.”

Quiver Representations

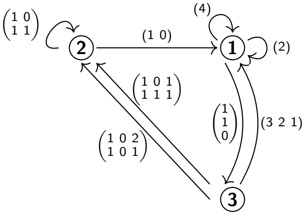
Def: A *representation* of $Q = (Q_0, Q_1)$ assigns ...

- a coordinate space $\mathbb{C}^{\gamma(i)}$ to each vertex i in Q_0, and
- a linear map $\mathbb{C}^{\gamma(i)} \to \mathbb{C}^{\gamma(j)}$ to each edge $i \to j$ in Q_1.



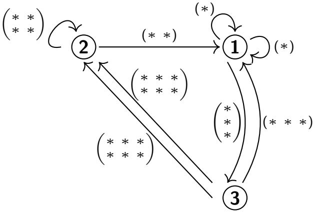
Dimension Vector

Every representation of Q has a *dimension vector* $\gamma : Q_0 \to \mathbb{Z}_{\geq 0}$.



Dimension Vector Moduli Space

Given $\gamma : Q_0 \to \mathbb{Z}_{\geq 0}$, let M_γ be the moduli space of all representations with dimension vector γ.



$$\mathsf{M}_\gamma \cong \prod_{i \to j} \mathbb{C}^{\gamma(i) \times \gamma(j)}$$

Equivariant cohomology

Obs: The moduli space M_γ carries an action of

$$\mathsf{G}_\gamma := \prod_{i \in Q_0} GL_{\gamma(i)}(\mathbb{C})$$

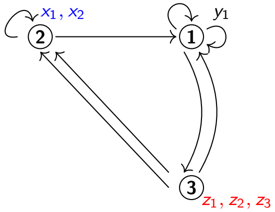
by change-of-basis at each vertex. We have the *equivariant cohomology ring*

$$\mathcal{H}_\gamma := H^\bullet_{\mathsf{G}_\gamma}(\mathsf{M}_\gamma)$$

with coefficients in $\mathbb{C}$.

Equivariant cohomology

Rmk: Can present $\mathcal{H}_\gamma$ using *polynomials*.



$$\mathcal{H}_\gamma = \mathbb{C}[x_1, x_2, y_1, z_1, z_2, z_3]^{S_2 \times S_1 \times S_3}$$

Cohomological Hall Algebra

Def: [Kontsevich-Soibelman] Let Q be a quiver. The *cohomological Hall algebra* of Q is

$$\mathcal{H} := \bigoplus_{\gamma} \mathcal{H}_\gamma$$

where γ ranges over all dimension vectors $Q_0 \to \mathbb{Z}_{\geq 0}$.

Multiplication: $\mathcal{H}_{\gamma_1} \otimes \mathcal{H}_{\gamma_2} \longrightarrow \mathcal{H}_{\gamma_1+\gamma_2}$

- Take direct sums of linear maps along each edge.
- Project along $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mapsto \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$ at each vertex.
- Apply pushforward maps.

$$H^\bullet_{G_{\gamma_1} \times G_{\gamma_2}}(\mathrm{M}_{\gamma_1} \times \mathrm{M}_{\gamma_2}) \xrightarrow{\sim} H^\bullet_{G_{\gamma_1,\gamma_2}}(M_{\gamma_1,\gamma_2}) \to H^\bullet_{G_{\gamma_1,\gamma_2}}(M_\gamma) \to H^\bullet_{G_\gamma}(\mathrm{M}_\gamma)$$

Cohomological Hall Algebra

Def: [Kontsevich-Soibelman] Let Q be a quiver. The *cohomological Hall algebra* of Q is

$$\mathcal{H} := \bigoplus_{\gamma} \mathcal{H}_\gamma$$

with $\cdot$ product.

Rmk: For $f_1 \in \mathcal{H}_{\gamma_1}$ and $f_2 \in \mathcal{H}_{\gamma_2}$ with $\gamma = \gamma_1 + \gamma_2$ we have

$$f_1 \cdot f_2 = \Sigma_{\gamma_1,\gamma_2} \cdot \left[f_1((x'_{i,\alpha})) \cdot f_2((x''_{i,\alpha})) \frac{\prod_{i,j\in Q_0} \prod_{\alpha_1=1}^{\gamma_1(i)} \prod_{\alpha_2=1}^{\gamma_2(j)} (x''_{j,\alpha_2} - x'_{i,\alpha_1})^{a_{ij}}}{\prod_{i\in Q_0} \prod_{\alpha_1=1}^{\gamma_1(i)} \prod_{\alpha_2=1}^{\gamma_2(i)} (x''_{i,\alpha_2} - x'_{i,\alpha_1})} \right]$$

where a_{ij} counts arrows $i \to j$ and $\Sigma_{\gamma_1,\gamma_2} \in \mathbb{C}[S_\gamma]$ is a shuffling operator.

Cohomological Hall Algebra

Def: [Kontsevich-Soibelman] Let Q be a quiver. The *cohomological Hall algebra* of Q is

$$\mathcal{H} := \bigoplus_{\gamma} \mathcal{H}_{\gamma}$$

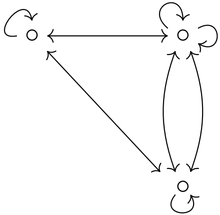
with $\cdot$ product.

Q: What does $\mathcal{H}$ look like?

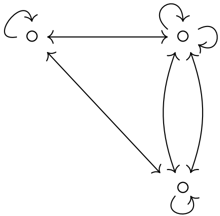
Nice Quivers

Def: Call a quiver Q *nice* if

- Q is connected
- Q is symmetric
- each vertex of Q has at least one loop.



Efimov's Theorem



Thm: [Efimov] Let Q be a nice quiver. After deforming the product $\cdot \rightsquigarrow \star$ with a sign twist, $\mathcal{H}$ is supercommutative and freely generated by a multigraded subspace $V = \bigoplus_{\gamma : Q_0 \to \mathbb{Z}_{\geq 0}} V_\gamma$. Each $V_\gamma = \bigoplus_k V_{\gamma,k}$ is a finite-dimensional singly-graded vector space.

DT invariants

Thm: [Efimov] Let Q be a nice quiver. The supercommutative algebra $\mathcal{H}$ is freely generated by a multigraded subspace $V = \bigoplus_{\gamma : Q_0 \to \mathbb{Z}_{\geq 0}} V_\gamma$. Each $V_\gamma = \bigoplus_k V_{\gamma,k}$ is a finite-dimensional singly-graded vector space.

Def: If $\gamma : Q_0 \to \mathbb{Z}_{\geq 0}$ is a dimension vector, the *numerical* and *quantum Donaldson-Thomas invariants* are

$$\mathrm{DT}_\gamma := \dim V_\gamma \qquad \mathrm{DT}_\gamma(q) := \sum_k \dim V_{\gamma,k} \cdot q^k$$

Q: How to compute $\mathrm{DT}_\gamma(q)$?

DT invariants

Thm: [Efimov] For Q nice, $\mathcal{H}$ freely generated by $V = \bigoplus_\gamma V_\gamma$.

$$\mathrm{DT}_\gamma := \dim V_\gamma \qquad \mathrm{DT}_\gamma(q) := \sum_k \dim V_{\gamma,k} \cdot q^k.$$

Cor: We have the power series identity

$$\sum_\gamma \frac{(-q^{-1/2})^{\chi_Q(\gamma,\gamma)} \cdot \mathbf{x}^\gamma}{\prod_{i\in Q_0}(1-q)(1-q^2)\cdots(1-q^{\gamma(i)})} =$$

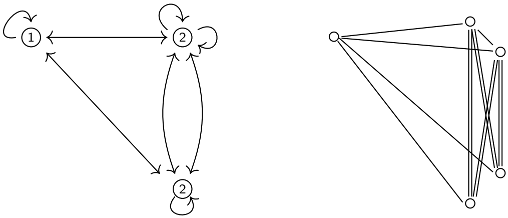
$$\mathrm{Exp}\left[\frac{1}{1-q}\sum_\gamma (-1)^{\chi_Q(\gamma,\gamma)} \mathrm{DT}_\gamma(q^{-1}) \cdot \mathbf{x}^\gamma\right]$$

where $\chi_Q(-,-)$ is the *Euler form* of Q and Exp is the *plethystic exponential*.

Nice Quivers to Graphs

Def: Given a nice quiver Q and a positive dimension vector $\gamma : Q_0 \to \mathbb{Z}_{>0}$, let G_γ be the graph with

- a family $\mathcal{F}_i$ of $\gamma(i)$ vertices for each $i \in Q_0$,
- $d - 1$ edges between each of the vertices in $\mathcal{F}_i$ if there are d loops at i in Q, and
- an edge between each vertex in $\mathcal{F}_i$ and $\mathcal{F}_j$ for all pairs $i \leftrightarrow j$ of arrows with $i \neq j$.



Break Divisors

Def: An *(effective) divisor* D on a graph G is a function $D : V(G) \to \mathbb{Z}_{\geq 0}$. The *degree* is $\deg(D) := \sum_{i \in V(G)} D(i)$.

Def: The *genus* of a connected graph G is

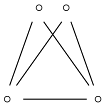
$$g(G) := |E(G)| - |V(G)| + 1.$$

Def: A divisor D on a connected graph G is a *break divisor* if

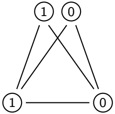
- $\deg(D) = g(G)$, and
- for all connected subgraphs $H \subseteq G$ we have $\deg(D\mid_H) \geq g(H)$.

Break Divisors

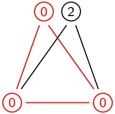
Def: A divisor D on G is a *break divisor* if $\deg(D) = g(G)$ and $\deg(D\mid_H) \geq g(H)$ for all connected $H \subseteq G$.



$g(G) = 2$



break



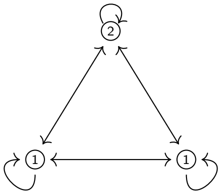
not break

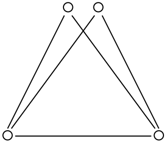
Nice Quivers to Polytopes

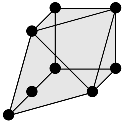
Def: A divisor $D : V(G) \to \mathbb{Z}_{\geq 0}$ is a *break divisor* if

- $\deg(D) = g(G)$ and
- $\deg(D\mid_H) \geq g(H)$ for all connected $H \subseteq G$.

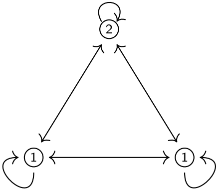
Def: If Q is a nice quiver and $\gamma : Q_0 \to \mathbb{Z}_{>0}$ is a dimension vector, let Z_γ be the locus of break divisors of G_γ.

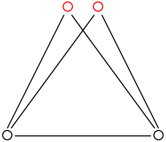


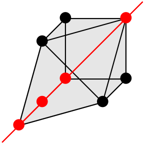




Nice Quivers to Polytopes

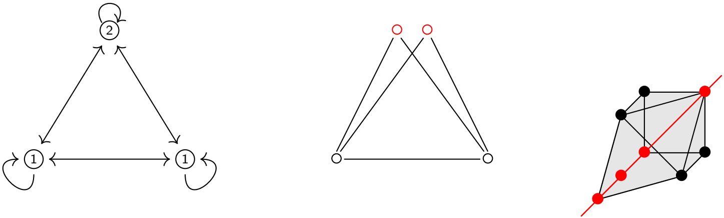






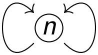
Obs: Z_γ carries an action of the symmetry group of G_γ, and in particular of $S_\gamma = \prod_{i \in Q_0} S_{\gamma(i)}$.

Main Theorem: Numerical



Thm: [RRT] The numerical DT invariant DT_γ is the number of S_γ-orbits in Z_γ.

Two-Loop Case



Fact: We have $Z_n = \mathbb{Z}^n \cap P_n$ where

$$P_n := \mathrm{conv}(w \cdot (n-2, n-3, \ldots, 1, 0, 0) \;:\; w \in S_n)$$

is the *trimmed permutohedron*. The locus Z_n has size n^{n-2}.

Thm: [Konvalinka-Tewari] The number of S_n-orbits in Z_n is

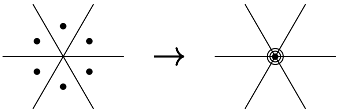
$$\frac{1}{n^2} \sum_{d|n} (-1)^{n+d} \mu(d) \binom{2d-1}{d}$$

(This sequence starts $1, 1, 1, 2, 5, 13, 35, 100, 300, 925, 2915 \ldots$)

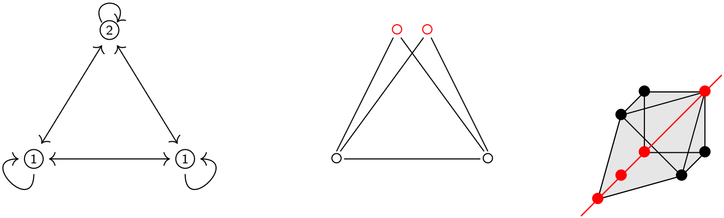
Orbit Harmonics

Let $Z \subseteq \mathbb{C}^n$ be a finite point locus. We obtain a *graded quotient ring* $R(Z)$ as follows.

$$\begin{aligned} Z &\rightsquigarrow \mathbf{I}(Z) = \{f \in \mathbb{C}[x_1, \dots, x_n] \, : \, f(\mathbf{z}) = 0 \text{ for all } \mathbf{z} \in Z\} \\ &\rightsquigarrow \operatorname{gr} \mathbf{I}(Z) \quad \text{(associated graded ideal in } \mathbb{C}[x_1, \dots, x_n]) \\ &\rightsquigarrow R(Z) := \mathbb{C}[x_1, \dots, x_n] / \operatorname{gr} \mathbf{I}(Z) \end{aligned}$$



Main Theorem: Quantum



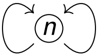
Thm: [RRT] Let $R(Z_\gamma)$ be the orbit harmonics quotient attached to Z_γ; it is a graded S_γ-module. The quantum DT invariant is the Hilbert series

$$\mathrm{DT}_\gamma(q) = \mathrm{Hilb}(R(Z_\gamma)^{S_\gamma}; q)$$

of its S_γ-fixed subspace.

Suggestion: Find the graded S_γ-structure of $R(Z_\gamma)$.

Two-Loop Case



Fact: We have $Z_n = \mathbb{Z}^n \cap P_n$ where

$$P_n := \mathrm{conv}(w \cdot (n-2, n-3, \ldots, 1, 0, 0) \,:\, w \in S_n)$$

is the *trimmed permutohedron*. The locus Z_n has size n^{n-2}.

Thm: [RRT; Berget-R] The restriction of $R(Z_n)$ from S_n to S_{n-1} is a graded refinement of the S_{n-1} action on length $n-1$ parking functions. The graded character is $(\omega \circ \mathrm{rev}_q)\nabla e_{n-1} \mid_{t \to 1}$.

Proof Ideas

Let Q be a nice quiver and $\gamma : Q_0 \to \mathbb{Z}_{>0}$ satisfy $\sum_{i \in Q_0} \gamma(i) = n$.

- Show that $R(Z_\gamma)$ is a quotient of $\mathbb{C}[x_1, \ldots, x_n]$ by a *power ideal* I_γ [Ardila-Postnikov, Postnikov-Shapiro].
- Express $R(Z_\gamma) \cong I_\gamma^\perp$ as a *Macaulay inverse system*; this is generated by *slim subgraph polynomials* in G_γ.
- De-symmetrize the constructions of Efimov to show that

  $$\mathrm{DT}_\gamma(q) = \mathrm{Hilb}((\mathbb{C}[x_1, \ldots, x_n]/J_\gamma)^{S_\gamma}; q)$$

  where J_γ is the *bond ideal* corresponding to G_γ.
- Show that the composite

  $$R(Z_\gamma) \cong I_\gamma^\perp \hookrightarrow \mathbb{C}[x_1, \ldots, x_n] \twoheadrightarrow \mathbb{C}[x_1, \ldots, x_n]/J_\gamma$$

  is a graded S_γ-module isomorphism.

Thanks for listening!!

- M. Reineke, B. Rhoades, and V. Tewari. Zonotopal algebras, orbit harmonics, and Donaldson-Thomas invariants of symmetric quivers. *Int. Math. Res. Not. IMRN*, 2023, rnad033.