Catalania

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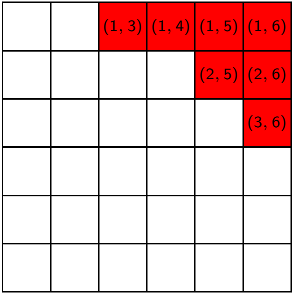
Root ideals

- Set of positive roots $R_+ = R_+(\mathrm{GL}_n) \stackrel{def}{=} \{(i,j) : 1 \le i < j \le n\}$.
- A *root ideal* $\Psi \subseteq R_+$ is an upper order ideal of positive roots.

Example. $\Psi = \{(1,3),(1,4),(1,5),(1,6),(2,5),(2,6),(3,6)\}$



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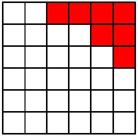
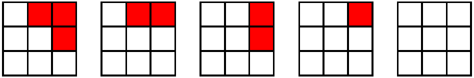


$\#$ of root ideals of $R_+(\mathrm{GL}_n)$ = the n-th Catalan number $\frac{1}{n+1}\binom{2n}{n}$.

Example. The 5 root ideals of $R_+(\mathrm{GL}_3)$:



Can it Catalanify?

What happens if we replace a product over positive roots with one over a root ideal?

- Weyl character formula $s_\mu(\mathbf{z}) = \sum_{w \in \mathcal{S}_n} w\Big(\mathbf{z}^\mu \prod_{1 \le i < j \le n} (1 - z_j/z_i)^{-1}\Big)$
- Modified Hall-Littlewoods $H_\mu(X;q) = \boldsymbol{\sigma}\Big(\mathbf{z}^\mu \prod_{i<j} (1 - q\, z_i/z_j)^{-1}\Big)$
- Hall-Littlewood polynomials
$$P_\mu(X;q) = \frac{1}{v_\mu(q)}\, \boldsymbol{\sigma}\Big(\mathbf{z}^\mu \prod_{i<j} (1 - q\, z_j/z_i)\Big)$$
- Cauchy formula for Schubert polynomials
$\prod_{1 \le i < j \le n} (x_i + y_j) = \sum_{w \in \mathcal{S}_n} \mathfrak{S}_w(x_1, \ldots, x_n) \mathfrak{S}_{ww_0}(y_n, \ldots, y_1)$
- Cauchy formula for keys $\prod_{1 \le i \le j \le n} \frac{1}{1 - x_i/y_j} = \sum_{\alpha \in \mathbb{N}^n} K_\alpha(\mathbf{x}) \hat{K}_{-\alpha}(\mathbf{y})$
- Cauchy formula for nonsymmetric Macdonald polynomials of Mimachi and Noumi
$$\prod_{j<i} \frac{(qt\, x_i y_j; q)_\infty}{(q x_i y_j; q)_\infty} \prod_{i<j} \frac{(t\, x_i y_j; q)_\infty}{(x_i y_j; q)_\infty} \prod_{i} \frac{(qt\, x_i y_i; q)_\infty}{(x_i y_i; q)_\infty} = \sum_{\alpha \in \mathbb{N}^n} a_\alpha E_\alpha(\mathbf{x}; q, t) E_\alpha(\mathbf{y}; q^{-1}, t^{-1})$$
- Identity of Littlewood and Schur $\sum_\lambda s_\lambda = \prod_i (1 - z_i)^{-1} \prod_{i<j} (1 - z_i z_j)^{-1}$

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Schur function straightening

- $\Lambda(X)$ = ring of symmetric functions in $X = x_1, x_2, \cdots$.
- $h_d(X) = \sum_{i_1 \leq \cdots \leq i_d} x_{i_1} \cdots x_{i_d}$ is the *homogeneous symmetric function*.
- Convention: $h_0 = 1$ and $h_d = 0$ for $d < 0$.

Def. Schur functions may be defined for any $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{Z}^n$ by

$$s_\gamma(X) = \det \left(h_{\gamma_i + j - i}(X) \right)_{1 \leq i,j \leq n} \in \Lambda(X).$$

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Proposition (Schur function straightening)

For $\gamma \in \mathbb{Z}^n$,

$$s_\gamma(X) = \begin{cases} \operatorname{sgn}(\gamma+\rho) s_{\mathrm{sort}(\gamma+\rho)-\rho}(X) & \text{if } \gamma+\rho \text{ has distinct nonnegative parts,} \\ 0 & \text{otherwise,} \end{cases}$$

- $\rho = (n-1, n-2, \ldots, 0)$,
- $\mathrm{sort}(\beta)$ = *weakly decreasing sequence obtained by sorting* β,
- $\operatorname{sgn}(\beta)$ = *sign of the shortest permutation taking* β *to* $\mathrm{sort}(\beta)$.

Example. $n = 4$, $\gamma = (3, 1, 2, 5)$.

$\gamma + \rho = (3, 1, 2, 5) + (3, 2, 1, 0) = (6, 3, 3, 5)$ has a repeated part.

Hence $s_{3125} = 0$.

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Example. $n = 4$, $\gamma = (4, 7, 1, 6)$.

$$\gamma + \rho = (4,7,1,6) + (3,2,1,0) = (7,9,2,6)$$

$$\mathrm{sort}(\gamma+\rho) = (9,7,6,2)$$

$$\mathrm{sort}(\gamma+\rho) - \rho = (6,5,5,2)$$

Hence $s_{4716} = s_{6552}$.

Catalan functions

Def. The *Weyl symmetrization operator* is the linear map determined by

$$\begin{aligned} \boldsymbol{\sigma} : \mathbb{Q}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}] &\to \Lambda(X), \\ \mathbf{z}^\gamma &\mapsto s_\gamma(X), \end{aligned}$$

where $\mathbf{z}^\gamma = z_1^{\gamma_1} \cdots z_n^{\gamma_n}$. Extends to a map from $\mathbb{Q}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}][[q]]$.

Def. (Chen-Haiman 2008, Panyushev 2010) Given a root ideal $\Psi \subseteq R_+$ and weight $\gamma \in \mathbb{Z}^n$, the associated *Catalan function* is

$$H_\gamma^\Psi(X;q) \overset{def}{=} \boldsymbol{\sigma}\Big(\mathbf{z}^\gamma \prod_{(i,j)\in\Psi} (1 - q z_i/z_j)^{-1}\Big).$$

Example. Let $\mu = (\mu_1, \ldots, \mu_n)$ be a partition.

- Empty root ideal: $H_\mu^\varnothing(X;q) = s_\mu(X)$.
- Full root ideal: $H_\mu^{R_+}(X;q) = H_\mu(X;q) = \sum_\lambda K_{\lambda\mu}(q) s_\lambda(X)$, the modified Hall-Littlewood polynomial.

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Example. For $\mu = 3321$, $\Psi = \{(1,3),(1,4),(2,4)\}$,

3		1, 3	1, 4
	3		2, 4
		2	
			1

$$H_\mu^\Psi(X;q)$$
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&H_\mu^\Psi(X;q) \\
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&= \boldsymbol{\sigma}\big((1 - q z_1/z_3)^{-1}(1 - q z_2/z_4)^{-1}(1 - q z_1/z_4)^{-1}\mathbf{z}^{3321} \big)
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&= \boldsymbol{\sigma}\big(\mathbf{z}^{3321} + q(\mathbf{z}^{3420} + \mathbf{z}^{4311} + \mathbf{z}^{4320}) + q^2(\mathbf{z}^{4410} + \mathbf{z}^{5301} + \mathbf{z}^{5310}) \\
&\quad + q^3(\mathbf{z}^{63-11} + \mathbf{z}^{5400} + \mathbf{z}^{6300}) + q^4(\mathbf{z}^{64-10} + \mathbf{z}^{73-10})\big)
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&\quad + q^3(\mathbf{z}^{63-11}+\mathbf{z}^{5400}+\mathbf{z}^{6300}) + q^4(\mathbf{z}^{64-10}+\mathbf{z}^{73-10})\big) \\
&= s_{3321} + q(s_{432}+s_{4311}) + q^2(s_{441}+s_{531}) + q^3 s_{54}.
\end{aligned}
$$

Modified Hall-Littlewood polynomials $H_\mu(X;q)$

$H_\mu(X;1) = h_\mu(X)$, the homogeneous symmetric function.

Schur expansion coefficients $K_{\lambda,\mu}(q)$ are the *Kostka-Foulkes polynomials*:

$$H_\mu(X;q) = \sum_\lambda K_{\lambda,\mu}(q) s_\lambda(X).$$

- The $K_{\lambda\mu}(q)$ originated in the character theory of $\mathrm{GL}_n(\mathbb{F}_q)$.
- The $K_{\lambda\mu}(q)$ are certain affine Kazhdan-Lusztig polynomials.
- The $H_\mu(X;q)$ are the graded Frobenius series of certain quotients of the *ring of coinvariants* $\mathbb{C}[y_1,\ldots,y_n]/(e_1,\ldots,e_n)$.

Theorem (Lascoux-Schützenberger 1978)

For any partition μ,

$$H_\mu(X;t) = \sum_T t^{charge(T)} s_{shape(T)}(X),$$

the sum over semistandard tableaux T of content μ.

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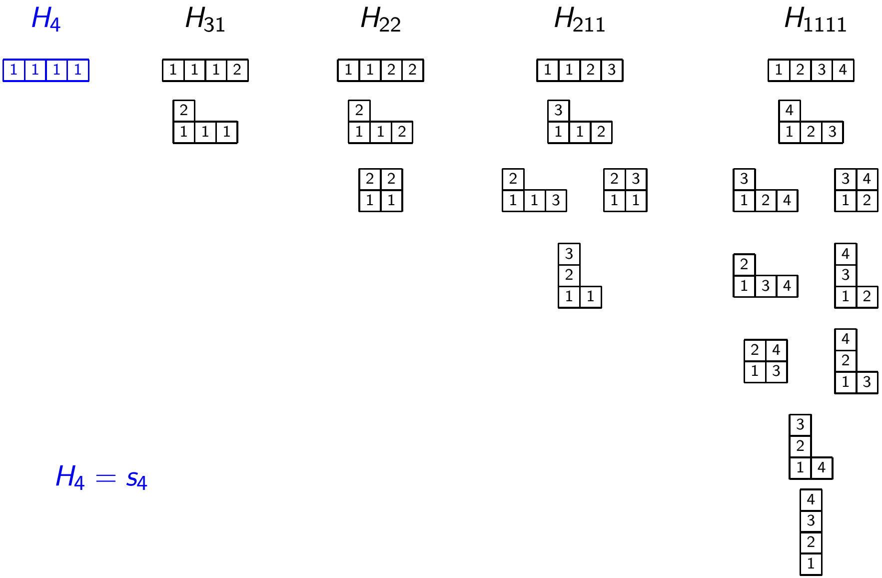
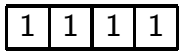
Modified Hall-Littlewood polynomials



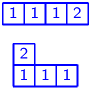
$H_4 = s_4$

Modified Hall-Littlewood polynomials

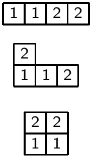
H_4



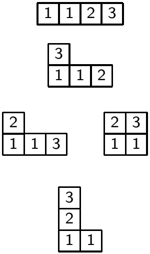
H_{31}



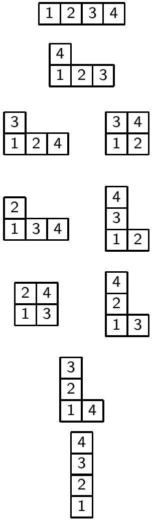
H_{22}



H_{211}

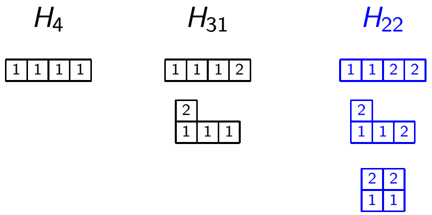


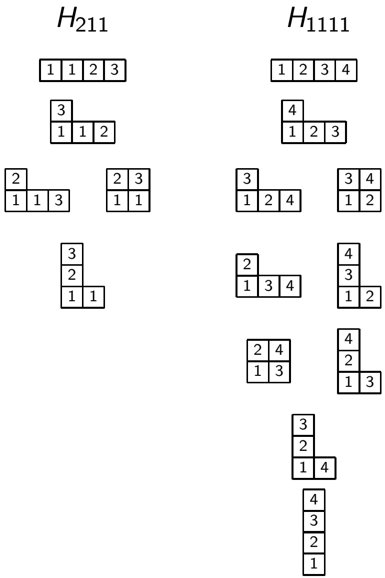
H_{1111}



$$H_{31} = s_{31} + q\,s_4$$

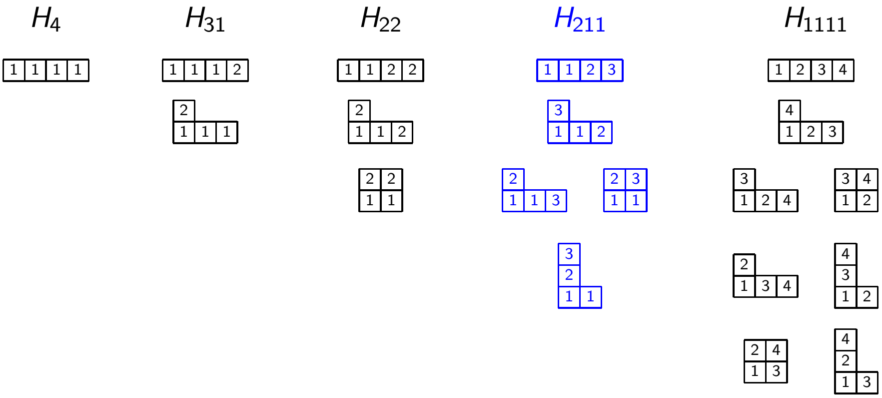
Modified Hall-Littlewood polynomials

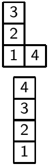




$$H_{22} = s_{22} + q\, s_{31} + q^2 s_4$$

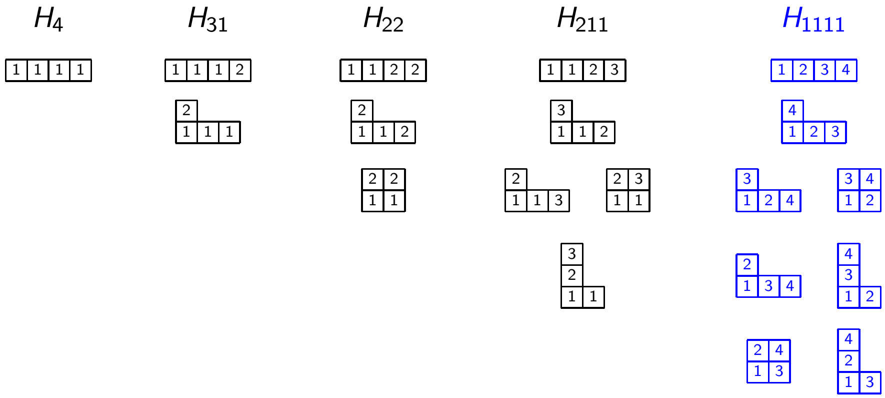
Modified Hall-Littlewood polynomials

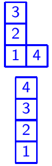




$$H_{211} = s_{211} + q(s_{22} + s_{31}) + q^2 s_{31} + q^3 s_4$$

Modified Hall-Littlewood polynomials





$$H_{1111} = s_{1111} + q\, s_{211} + q^2(s_{22} + s_{31}) + q^3(s_{31}+s_{211})+q^4(s_{31}+s_{22})+q^5 s_{31}+q^6 s_4$$

Catalan functions

Catalan functions

- contain the modified Hall-Littlewood polynomials $H_\mu(X; q)$.
- contain the parabolic Hall-Littlewood polynomials studied by A. N. Kirillov-Schilling-Shimozono, Schilling-Warnaar, Shimozono, Shimozono-Weyman, Shimozono-Zabrocki around 2000.
- contain the k-Schur functions $s_\mu^{(k)}(X; q)$.
- are graded Euler characteristics of vector bundles on the flag variety. Cohomology vanishing results proven by Broer (1994), Hague (2009), Panyushev (2010).
- are Schur positive for partition μ, conjectured by Chen-Haiman (2008), proven by B.-Morse-Pun (2020).

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k-Schur functions

The k-Schur functions $s_\mu^{(k)}(X;q)$ arose in the study of Lapointe-Lascoux-Morse (2000) on the Macdonald positivity conjecture.

k-Schur functions $\left\{s_\mu^{(k)}(X;1)\right\}_{\mu_1 \le k}$ form a basis for $\mathbb{Q}[h_1, \ldots, h_k] \subset \Lambda(X)$.

Lam (2008) and Lapointe-Morse (2007) connected them to affine Schubert calculus. At $q = 1$, they represent Schubert classes in the homology of the affine Grassmannian $\mathrm{Gr}_{SL_{k+1}}$.

k-Schur Catalan functions

Def. For a partition μ of length $\leq n$ and with $\mu_1 \leq k$, define the root ideal

$$\Delta^k(\mu) = \big\{(i,j) \in R_+(\mathrm{GL}_n) : k - \mu_i + i < j\big\}.$$

$$\text{“}(\#\text{non-roots in row } i) + \mu_i = k\text{”}$$

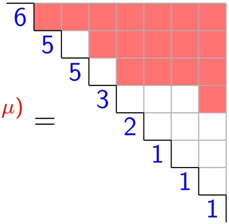
Theorem (B.-Morse-Pun-Summers 2018)

The k-Schur functions have the following description as Catalan functions:

$$s^{(k)}_{\mu}(X;q) = H^{\Delta^k(\mu)}_{\mu}(X;q).$$

Example. For $k = 6$ and $\mu = 65532111$,

$$s^{(k)}_{\mu}(X;q) = H^{\Delta^k(\mu)}_{\mu} =$$



Positive Branching

Theorem (B.-Morse-Pun-Summers 2018)

The k-Schur functions $s_\mu^{(k)}(X;q)$ satisfy

(shift invariance) $\quad e_n^\perp s_{\mu+1^n}^{(k+1)} = s_\mu^{(k)} \quad$ *for* $\ell(\mu) \leq n$,

(positive branching) $\quad s_\mu^{(k)} = \sum_\lambda a_{\lambda\mu}(q)\, s_\lambda^{(k+1)} \quad$ *with* $a_{\lambda\mu}(q) \in \mathbb{N}[q]$,

(Schur positivity) $\quad s_\mu^{(k)} = \sum_\lambda c_{\lambda\mu}(q)\, s_\lambda \quad$ *with* $c_{\lambda\mu}(q) \in \mathbb{N}[q]$.

Goal: Given a class of symmetric functions, realize it as a subfamily of Catalan functions or variations of Catalan functions.

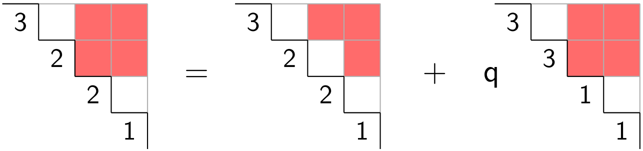
Realizing such a class as part of the larger set of Catalan functions provides stepping stones for an inductive approach to Schur positivity.

Root expansion

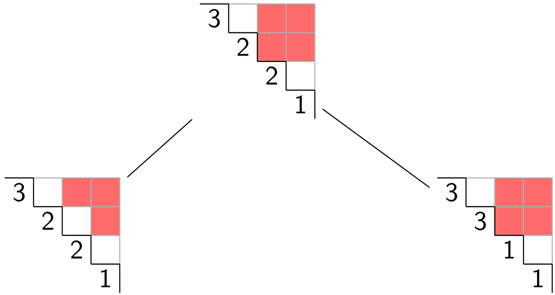
Prop. Suppose that $\alpha \in \Psi$ with $\Psi \setminus \alpha$ a root ideal. Then

$$H_\mu^{\Psi} = H_\mu^{\Psi \setminus \alpha} + q\, H^{\Psi}_{\mu+\epsilon_i-\epsilon_j}.$$

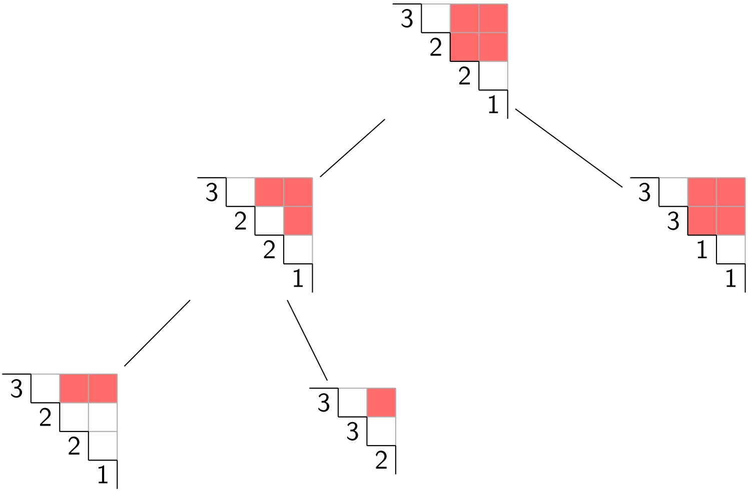
Example. For $\alpha = (2,3)$,



Root expansion tree



Root expansion tree



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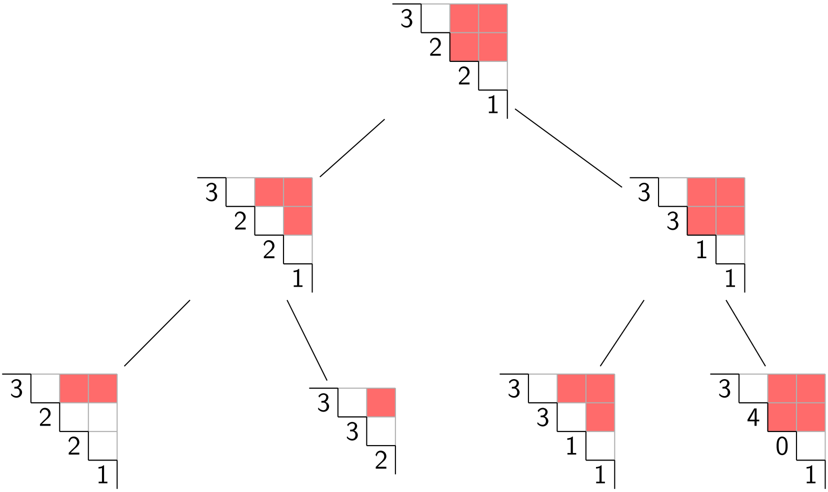
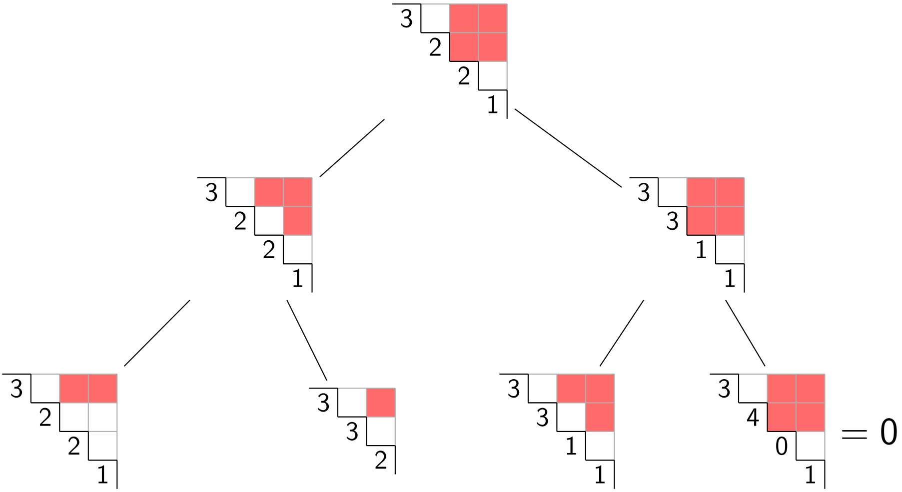


Root expansion tree



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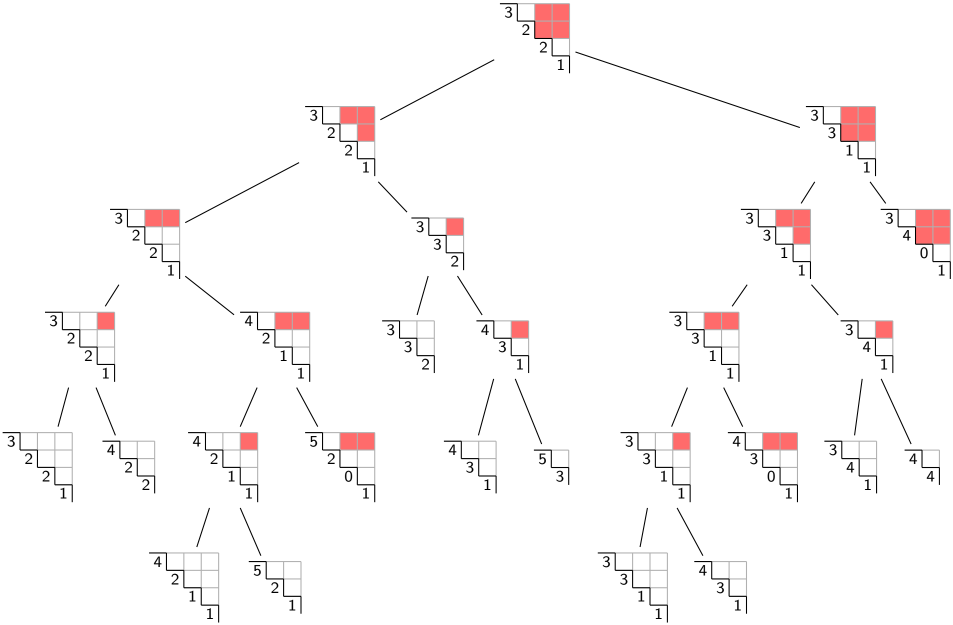
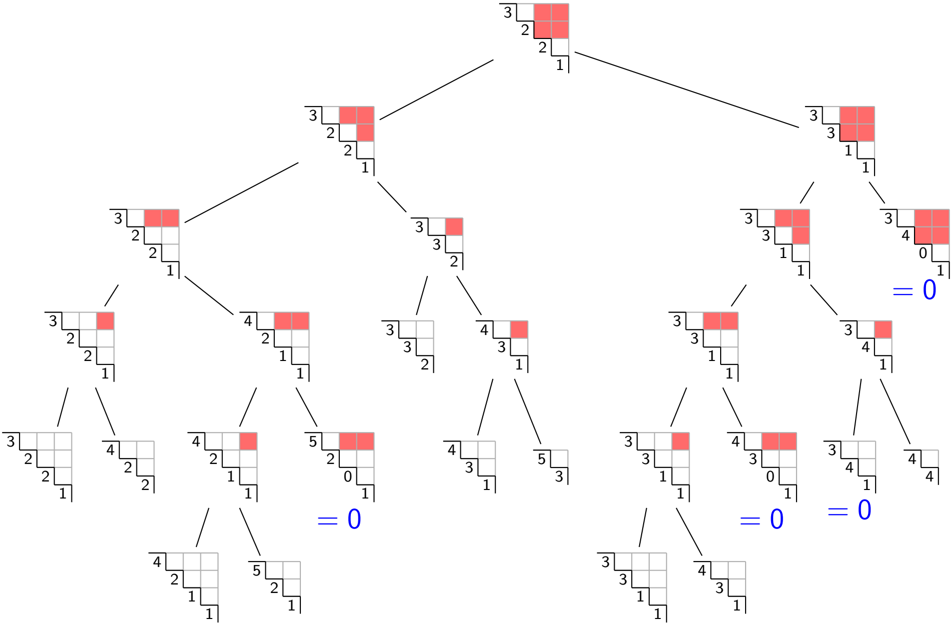


Root expansion tree



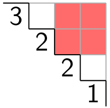
A Littlewood-Richardson rule via root expansion

Conjecture. Any Catalan function of partition weight has a root expansion tree with leaves which are 0 or single Schur functions.

Prop. For a rectangular root ideal $\Psi = \{(i,j) \in R_+ : i \leq d, j > d\}$,

$$H_\mu^\Psi(X;1) = s_{(\mu_1,\ldots,\mu_d)} s_{(\mu_{d+1},\ldots,\mu_n)}.$$

Example.



$$= \Big|_{q=1} \quad s_{32} s_{21}$$

Using root expansion, we can obtain a rule for $s_{(\mu_1,\ldots,\mu_d)} s_{(\mu_{d+1},\ldots,\mu_n)}$ when μ is a partition.

Root expansion and affine Bruhat order

Theorem

The k-Schur functions have the following description as Catalan functions:

$$s_\mu^{(k)}(X;q) = H_\mu^{\Delta^k(\mu)}(X;q).$$

Proof idea:

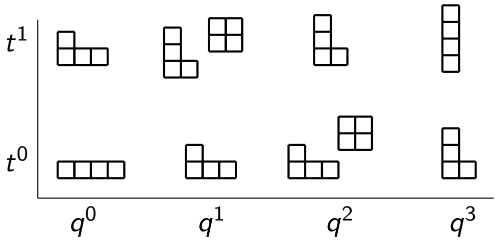
- Show that the k-Schur Catalan functions $H_\mu^{\Delta^k(\mu)}(X;q)$ satisfy a dual Pieri rule.
- Show using root expansion that $H_{\mu-\epsilon_i}^{\Delta^k(\mu-\epsilon_i)}(X;q)$ straightens to 0 or a single k-Schur Catalan function $H_\lambda^{\Delta^k(\lambda)}(X;q)$ for partition λ.
- Expanding to the class of Catalan functions yields lots of intermediate objects!
- Combinatorics of the strong Bruhat order of the affine symmetric group is somehow encoded in the definition of $\Delta^k(\mu)$.

The *modified Macdonald polynomials* $\tilde{H}_\mu = \tilde{H}_\mu(X; q, t)$ are Schur positive symmetric functions over $\mathbb{Q}(q, t)$.

- Specialize to modified Hall-Littlewood polynomials: $\tilde{H}_\mu(X; 0, t) = H_\mu(X; t)$ and $\tilde{H}_\mu(X; q, 0) = H_{\mu'}(X; q)$.
- Bigraded Frobenius series of $\mathcal{S}_n$-submodules of $\mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ called Garsia-Haiman modules.

Example.

$$\tilde{H}_{31} = s_4 + (q + t + q^2)s_{31} + (qt + q^2)s_{22} + (qt + q^2 t + q^3)s_{211} + q^3 t\, s_{1111}$$



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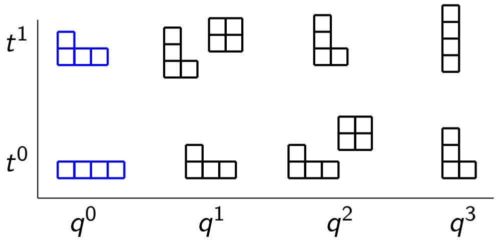
$\tilde{H}_{31}(X; 0, t) = H_{31}(X; t)$



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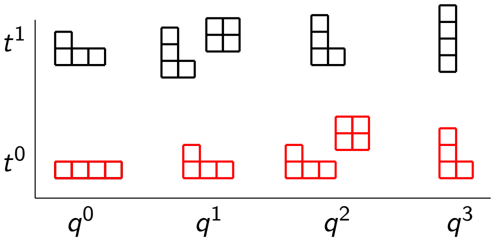

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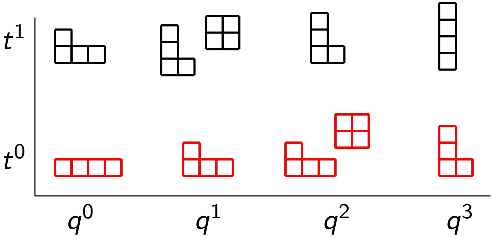


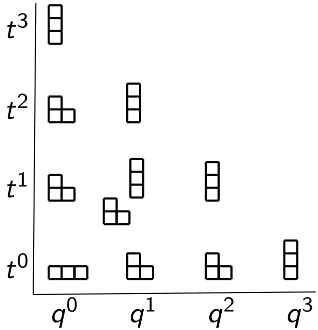
The nabla operator

Def. ∇ is the linear operator on symmetric functions satisfying $\nabla \tilde{H}_\mu = t^{n(\mu)} q^{n(\mu')} \tilde{H}_\mu$, where $n(\mu) = \sum_i (i-1)\mu_i$.

- ∇e_n is the bigraded Frobenius series of the *ring of diagonal coinvariants* $\mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]/\mathcal{I}$, where $\mathcal{I}$ is the ideal generated by $\mathcal{S}_n$-invariant polynomials with no constant term.

Example.

$$\nabla e_3 = s_3 + (q + t + q^2 + qt + t^2)s_{21} + (qt + q^3 + q^2 t + qt^2 + t^3)s_{111}$$



Positivity and representation theory related to nabla

	Schur positive	combinatorial formula Conjecture/Proved	representation theory
∇e_k Shuffle conj/thm	✓	HHLRU 05 Carlsson-Mellit 15	 Haiman 02
$\nabla^m e_k$	✓	HHLRU 05 Mellit 16	 Haiman 02
$\mathbf{e}_{m,n} \cdot (-\mathbf{1}^{n+1})$	✓	Gorsky-Negut 13	Hikita 12
$\mathbf{e}_{km,kn} \cdot (-\mathbf{1}^{k(n+1)})$ km, kn-shuffle	✓	Bergeron-Garsia-Leven-Xin 14 Mellit 16	
∇H_λ compositional shuffle	✓	Haglund-Morse-Zabrocki 10 Carlsson-Mellit 15	
$\pm\nabla p_k$ square paths	✓	Loehr-Warrington 07 Sergel 16	
$\pm\nabla m_\lambda$	conj	Sergel 18 (hook λ)	
$\pm\nabla s_\lambda$	✓	Loehr-Warrington 07 BHMPS 21	

Building off work by Armstrong, Can-Loehr, Egge-Haglund-Kremer-Killpatrick, Garsia-Haglund, Garsia-Xin-Zabrocki, Hicks, Lenart, Loehr-Remmel, and many others.

Def. The *Catalanimal* indexed by $R_q, R_t, R_{qt} \subseteq R_+$ and $\mu \in \mathbb{Z}^n$ is

$$H(R_q, R_t, R_{qt}, \mu) = \boldsymbol{\sigma}\left(\frac{\mathbf{z}^{\mu} \prod_{(i,j)\in R_{qt}} \left(1 - qt\, z_i/z_j\right)}{\prod_{(i,j)\in R_q} \left(1 - q\, z_i/z_j\right) \prod_{(i,j)\in R_t} \left(1 - t\, z_i/z_j\right)} \right).$$

Catalanimals

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Example. With $n = 3$,

$$H(R_+, R_+, \{(1,3)\}, (111)) = \boldsymbol{\sigma}\left(\frac{\mathbf{z}^{111}(1 - qt\, z_1/z_3)}{\prod_{1\le i<j\le 3}(1 - q\, z_i/z_j)(1 - t\, z_i/z_j)} \right)$$

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$$= s_{111} + (q + t + q^2 + qt + t^2)s_{21} + (qt + q^3 + q^2 t + qt^2 + t^3)s_3$$

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$$= s_{111} + (q + t + q^2 + qt + t^2)s_{21} + (qt + q^3 + q^2 t + qt^2 + t^3)s_3$$

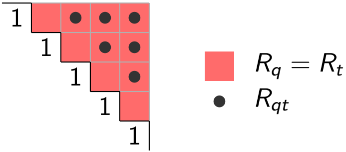
$$= \omega \nabla e_3.$$

The ∇e_n Catalanimal

Def. The *∇e_n Catalanimal* $H(R_q, R_t, R_{qt}, \mu)$ is given by

- $R_q = R_t = R_+$,
- $R_{qt} = \{(i,j) \in R_+ : i < j - 1\}$,
- weight $\mu = 1^n$.

Example. The ∇e_n Catalanimal for $n = 5$:



Building off work of Negut and Schiffmann-Vasserot on the shuffle algebra,

Theorem (B.-Haiman-Morse-Pun-Seelinger 2021)

$$\omega \nabla e_n = H(R_+, R_+, R_{qt}, 1^n) = \boldsymbol{\sigma}\left(\frac{z_1 \cdots z_n \prod_{i<j-1} (1 - q\, t\, z_i/z_j)}{\prod_{i<j} (1 - q\, z_i/z_j) \prod_{i<j} (1 - t\, z_i/z_j)} \right).$$

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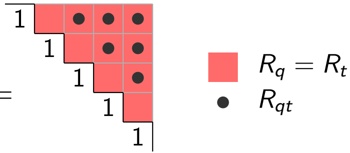
$\omega \nabla e_5 =$



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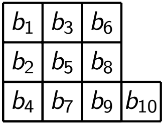
The ∇s_λ Catalanimal

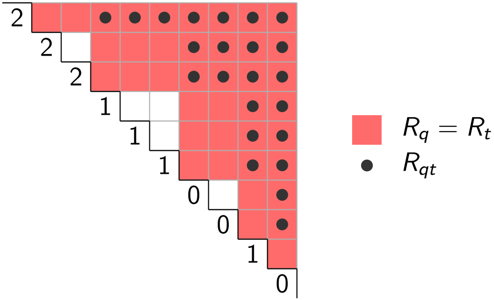
Def. For partition λ, define the ∇s_λ *Catalanimal* $H(R_q, R_t, R_{qt}, \mu)$ by

- $R_+ \supseteq R_q = R_t \supseteq R_{qt}$,
- $R_+ \setminus R_q =$ pairs of boxes in the same diagonal,
- $R_q \setminus R_{qt} =$ pairs going between adjacent diagonals,
- μ: fill each diagonal D of λ with
  $1 + \chi(D \text{ contains a row start}) - \chi(D \text{ contains a row end})$.
  Listing this filling in diagonal reading order gives μ.

Example. The ∇s_λ Catalanimal for $\lambda = 433$:



$\lambda = (433)$



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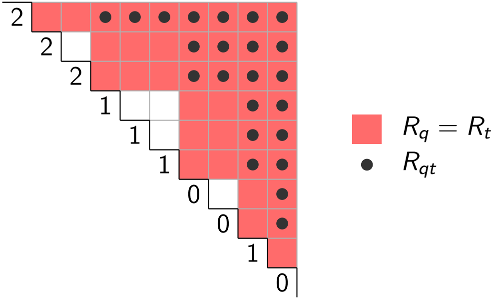
Example. The ∇s_λ Catalanimal for $\lambda = 433$:

2	2	1	
2	1	0	
1	0	1	0

μ, as a filling of λ



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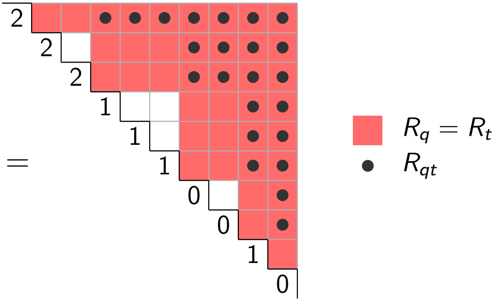
Example. The ∇s_λ Catalanimal for $\lambda = 433$:

$$(qt)^{-9}\omega\nabla s_\lambda \quad =$$



Theorem (B.-Haiman-Morse-Pun-Seelinger 2021)

For a partition λ, let $H(R_q, R_t, R_{qt}, \mu)$ be the Catalanimal constructed above. Then for some $c_\lambda \in \pm q^{\mathbb{Z}} t^{\mathbb{Z}}$,

$$c_\lambda\, \omega\nabla s_\lambda = H(R_q, R_t, R_{qt}, \mu) = \boldsymbol{\sigma}\left(\frac{\mathbf{z}^\mu \prod_{(i,j)\in R_{qt}} \left(1 - q\,t\,\frac{z_i}{z_j}\right)}{\prod_{(i,j)\in R_q}\left(1 - q\,\frac{z_i}{z_j}\right)\prod_{(i,j)\in R_t}\left(1 - t\,\frac{z_i}{z_j}\right)}\right).$$

Results arising from Catalanimal formulas

- New proof of the shuffle theorem.
- A shuffle theorem for paths under any line.
- Proof of the Loehr-Warrington conjecture,
  a Schur positive formula for ∇s_λ.
- Proof of the extended Delta conjecture of Haglund-Remmel-Wilson.
- A Catalan-style formula for the modified Macdonald polynomials.
- A connection between Catalanimals and the shuffle algebra.

Catalan-style formulas

We have obtained Catalan-style formulas for

- k-Schur functions $s_\mu^{(k)}(X;q) = H_\mu^{\Delta^k(\mu)}(X;q)$.
- K-theoretic k-Schur functions.
- ∇e_n, ∇s_λ, ∇H_λ, and ∇(LLT polynomial).
- $\Delta_{h_\ell} \Delta'_{e_k} e_n$ from the extended Delta conjecture.
- Modified Macdonald polynomials $\tilde{H}_\mu(X;q,t)$.

Catalania

Research directions:

(1) Find Catalan-style formulas for a known class of polynomials.
(2) Study the broader class of functions uncovered in (1).
(3) Use Catalan-style formulas to prove positivity.
   - Develop root expansion techniques.
   - Find Cauchy formulas for expanding Catalan-style formulas.

Can it Catalanify?

What happens if we replace a product over positive roots with one over a root ideal?

- Weyl character formula $s_\mu(\mathbf{z}) = \sum_{w \in \mathcal{S}_n} w\Big(\mathbf{z}^\mu \prod_{1 \le i < j \le n} (1 - z_j/z_i)^{-1}\Big)$
- Modified Hall-Littlewoods $H_\mu(X;q) = \boldsymbol{\sigma}\Big(\mathbf{z}^\mu \prod_{i<j} (1 - q\, z_i/z_j)^{-1}\Big)$
- Hall-Littlewood polynomials
$$P_\mu(X;q) = \frac{1}{v_\mu(q)} \boldsymbol{\sigma}\Big(\mathbf{z}^\mu \prod_{i<j} (1 - q\, z_j/z_i)\Big)$$
- Cauchy formula for Schubert polynomials
$\prod_{1 \le i < j \le n} (x_i + y_j) = \sum_{w \in \mathcal{S}_n} \mathfrak{S}_w(x_1, \dots, x_n) \mathfrak{S}_{ww_0}(y_n, \dots, y_1)$
- Cauchy formula for keys $\prod_{1 \le i \le j \le n} \frac{1}{1 - x_i/y_j} = \sum_{\alpha \in \mathbb{N}^n} K_\alpha(\mathbf{x}) \hat{K}_{-\alpha}(\mathbf{y})$
- Cauchy formula for nonsymmetric Macdonald polynomials of Mimachi and Noumi
$$\prod_{j<i} \frac{(qt\, x_i y_j; q)_\infty}{(q\, x_i y_j; q)_\infty} \prod_{i<j} \frac{(t\, x_i y_j; q)_\infty}{(x_i y_j; q)_\infty} \prod_{i} \frac{(qt\, x_i y_i; q)_\infty}{(x_i y_i; q)_\infty} = \sum_{\alpha \in \mathbb{N}^n} a_\alpha E_\alpha(\mathbf{x}; q, t) E_\alpha(\mathbf{y}; q^{-1}, t^{-1})$$
- Identity of Littlewood and Schur $\sum_\lambda s_\lambda = \prod_i (1 - z_i)^{-1} \prod_{i<j} (1 - z_i z_j)^{-1}$

K-theory

Def. The *dual stable Grothendieck polynomials* indexed by $\gamma \in \mathbb{Z}^n$ is

$$g_\gamma(X) = \det\left(h^{(i-1)}_{\gamma_i+j-i}(X)\right)_{1\le i,j\le n} \in \Lambda(X),$$

where $h^{(r)}_m = h_m(\underbrace{1,1,\ldots,1}_{r\text{ 1's}}, x_1, x_2, \ldots)$.

- $g_\lambda = s_\lambda +$ lower degree terms.
- The basis $\{g_\mu\}_{\text{partitions } \mu}$ is dual to the basis of stable Grothendieck polynomials $\{G_\lambda\}_{\text{partitions } \lambda}$ under the Hall inner product.
- G_λ's represent Schubert classes in the K-theory of the Grassmannian.
- G_λ's have a formula in terms of set valued tableaux.

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K-theoretic Catalan functions

Def. (B.-Morse-Seelinger 2020) The *Katalan function* indexed by root ideals $\Psi, \mathcal{L} \subseteq R_+$ and weight $\gamma \in \mathbb{Z}^n$ is

$$K_\gamma^{\Psi,\mathcal{L}}(X) \stackrel{def}{=} \boldsymbol{\sigma}_g\left(\frac{\prod_{(i,j)\in\mathcal{L}}(1-1/z_j)\,\mathbf{z}^\gamma}{\prod_{(i,j)\in\Psi}(1-z_i/z_j)}\right),$$

where $\boldsymbol{\sigma}_g : \mathbf{z}^\alpha \mapsto g_\gamma(X)$.

Example. For $\mu = 3321$ and $\Psi, \mathcal{L}$ as shown,

$K_\mu^{\Psi,\mathcal{L}} =$

$$= \boldsymbol{\sigma}_g\left(\frac{(1-1/z_4)^2\,\mathbf{z}^{3321}}{(1-z_1/z_3)(1-z_1/z_4)(1-z_2/z_4)}\right)$$

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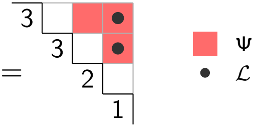
$K_\mu^{\Psi,\mathcal{L}} =$ 

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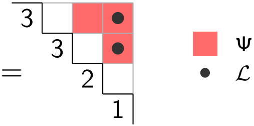
$K_\mu^{\Psi,\mathcal{L}} =$ 

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K-theoretic k-Schur functions

Def. The *K-k-Schur functions* $g_\mu^{(k)}$ are Schubert representatives for the K-homology of the affine Grassmannian $\mathrm{Gr}_{SL_{k+1}}$.

- Studied by Lam-Schilling-Shimozono (2010), Morse (2012), Ikeda-Iwao-Maeno (2018), Takigiku (2019).

Theorem (B.-Morse-Seelinger 2020)

The K-k-Schur functions are a subfamily of Katalan functions:

$$g_\mu^{(k)} = K_\mu^{\Delta^k(\mu), \Delta^{k+1}(\mu)}.$$

Example. For $k = 6$ and $\mu = 65532111$,

$g_\mu^{(k)} =$

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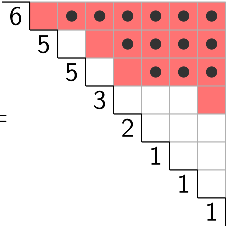

■ $\Psi = \Delta^k(\mu)$

• $\mathcal{L} = \Delta^{k+1}(\mu)$

Branching Positivity

Theorem (B.-Morse-Seelinger 2020)

The K-k-Schur functions $g_\mu^{(k)}(X)$ satisfy

(shift invariance) $G_{1^n}^{\perp}\, g_{\mu+1^n}^{(k+1)} = g_\mu^{(k)}$ *for* $\ell(\mu) \le n$,

(alternating branching) $g_\mu^{(k)} = \sum_\lambda a_{\lambda\mu}\, g_\lambda^{(k+1)}$ *with* $(-1)^{|\mu|-|\lambda|} a_{\lambda\mu} \in \mathbb{N}$.