

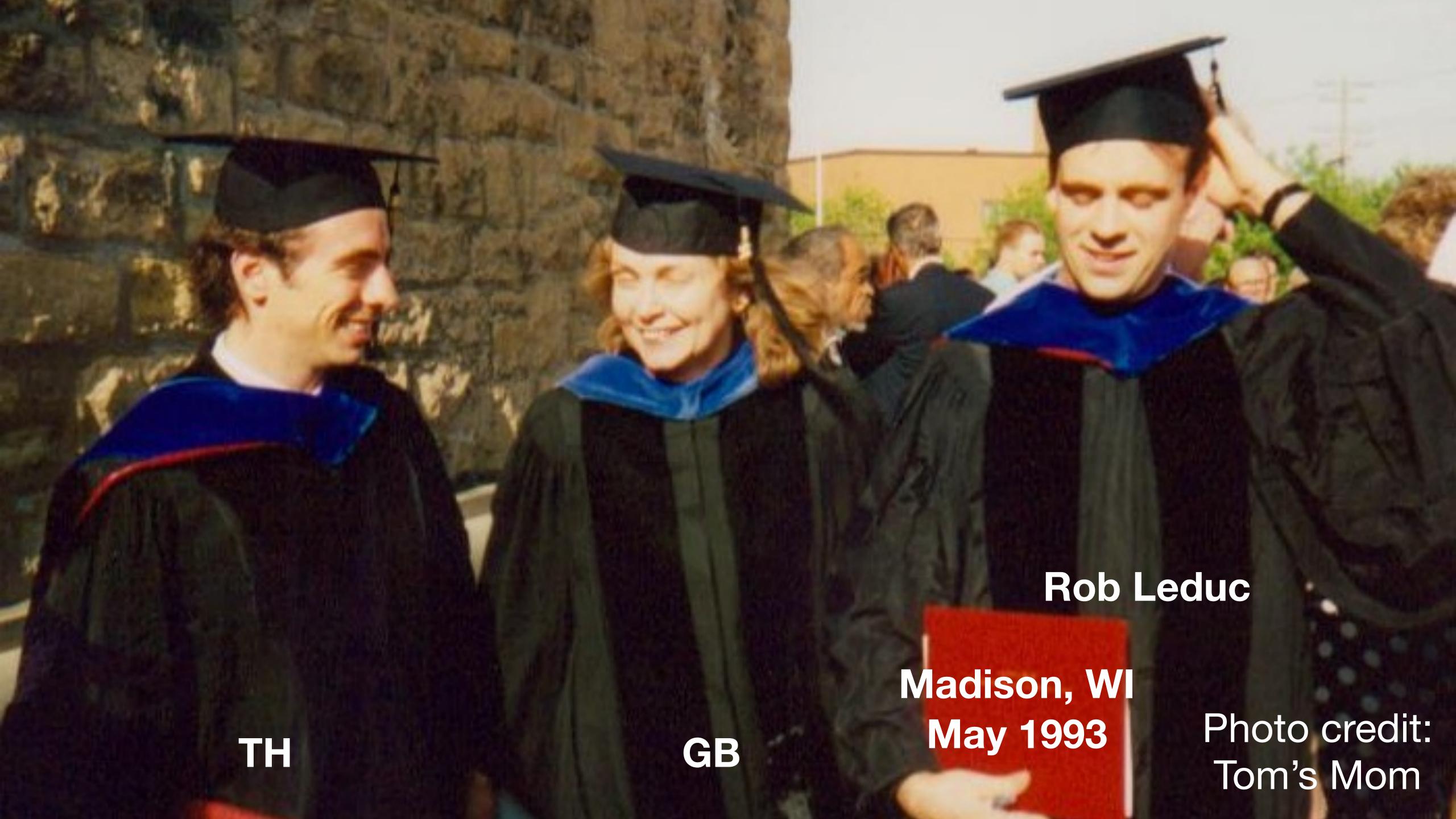
Georgia Benkart

FPSAC Memorial Lecture



Tom Halverson

Macalester College 17 July 2023





AWM Reception, JMM Baltimore 2014

March 2022

of the American Mathematical Society March 2022 Volume 69, Number 3

Gems from the Work of Georgia Benkart

Tom Halverson and Arun Ram

mathematics, beautiful style, and wonderful mathematical personality.

Classifying simple Lie algebras. In algebra in 1974, the

Classifying simple Lie algebras. In algebra in 1974, the air was thick with the classification of finite simple groups, with new finite simple groups being discovered in a frenzy, and the question always in the air:

"Have we found them all?"

At that time there was another such classification effort beginning: a search for all of the finite-dimensional simple Lie algebras.

In characteristic 0 the problem had been completed by Cartan and Killing around 1894, resulting in the list of Dynkin diagrams (Figure 1), which are in bijection with the finite-dimensional simple Lie algebras. Over an algebraically closed field of characteristic p > 7, four additional series occur:

- the Witt Lie algebras W(m, n),
- the special Lie algebras S(m, n)⁽¹⁾,
- the Hamiltonian Lie algebras $H(2m, \underline{n})^{(2)}$,
- the contact Lie algebras $K(2m+1,\underline{n})^{(1)}$.

The monograph by Benkart, Gregory, and Premet [BGP09] provides complete details on these algebras. They are known as the generalized Cartan-type Lie algebras, because they are derived from Cartan's four infinite families (Witt, special, Hamiltonian, contact) of *infinite-dimensional* complex Lie algebras. Cartan's work set the stage for Kostrikin–Šafarevič [KŠ66], who identified the above four unifying families of simple Lie algebras living in the Witt algebras. Earlier work of George Seligman [Sel67] (also at Yale) emphasized the role and the importance of the Lie algebras of Cartan type. George was one of Jacobson's first students and Georgia was one of his last.

In 1966, Kostrikin and Šafarevič conjectured that the Cartan-type Lie algebras and the Lie algebras coming from characteristic 0 were *all* of the finite-dimensional simple Lie algebras (over an algebraically closed field) in characteristic *p*. The original formulation was for "restricted" Lie algebras, and the general statement for finite-dimensional simple Lie algebras is the "Generalized Kostrikin–Šafarevič conjecture."

MATHEMATICAL SOCIETY

375

Algebraic geometry Associative rings and algebras

Combinatorics Commutative rings and algebras Convex and discrete

geometry General topology Group theory and generalizations

History and biography Nonassociative rings and

algebras Order, lattices, ordered algebraic structures Othe

Probability theory and stochastic processes

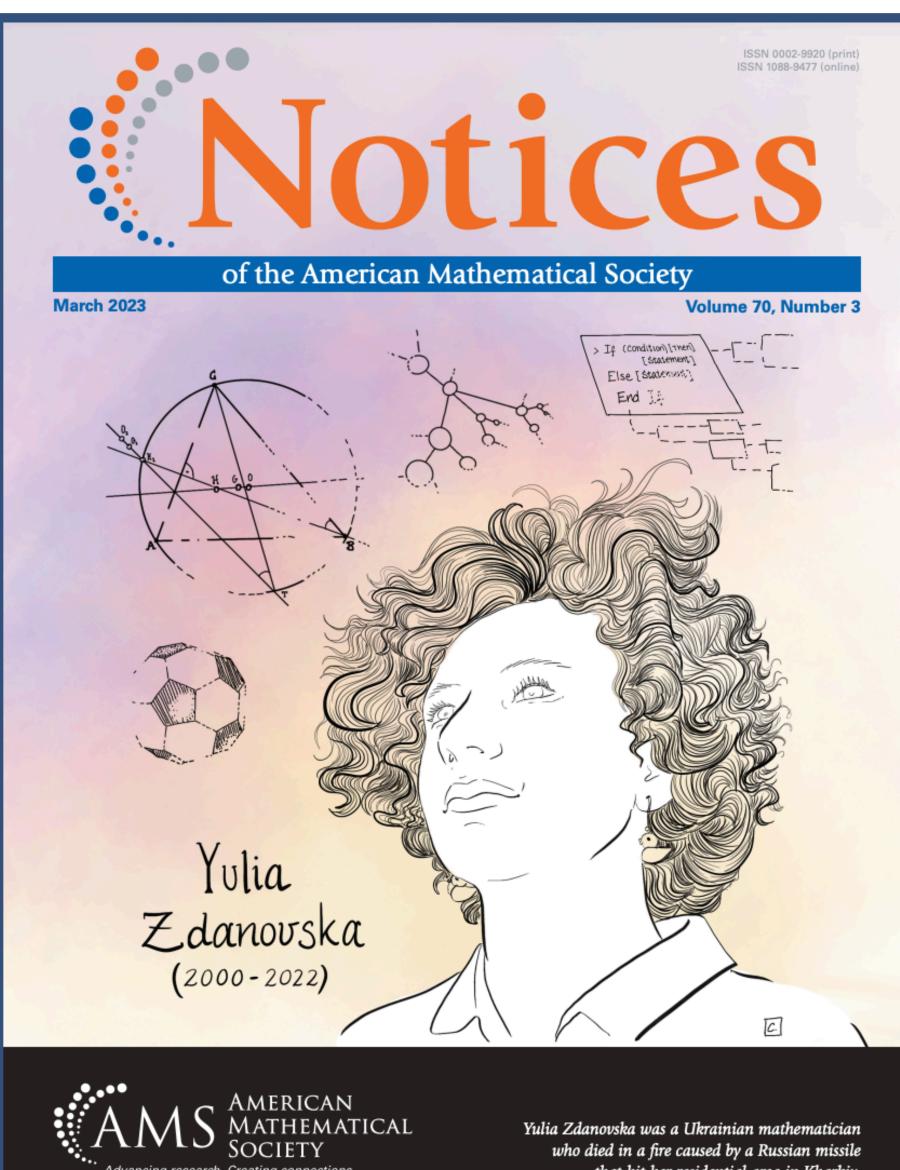
MathSciNet, 2023

Georgia Benkart completed her PhD in 1974 at Yale Uni-

HISTORY

- BA, Ohio State, 1970
- PhD, Yale, 1974,
 Nathan Jacobson
- UW Madison,Postdoc, 1974-76
- Assistant Professor, 1976-79
- Associate Professor, 1979-1983
- Professor, 1983-2007
- Professor Emeriti,2007-2022

March 2023





that hit her residential area in Kharkiv.

MEMORIAL TRIBUTE

Remembering Georgia Benkart

Alejandro Adem, Tom Halverson, Arun Ram, and Efim Zelmanov

Georgia Benkart passed away unexpectedly in Madison, Wisconsin, on April 29, 2022. Georgia earned her BA from Ohio State University and her PhD in 1974 from Yale University under Nathan Jacobson. She was a profoundly influential scholar and leader in the fields of Lie theory, representation theory, combinatorics, and noncommutative algebra. She spent her career at UW-Madison, where she was the second woman to join the department and the second to earn tenure. At the time of Georgia's retirement in 2006, she was the E. B. Van Vleck Professor of Mathematics. Georgia is survived by her sister, Paula Benkart, who also attended Ohio State University and earned a PhD in History from Johns Hopkins University in 1975.

Georgia was an inspiring teacher, the advisor to 22 PhD students, and a mentor to scores of mathematicians around the world. She published more than 130 articles and research monographs and gave more than 350 invited talks, including plenary lectures at AMS meetings, the AWM Noether Lecture at the Joint Mathematics Meetings, and the Emmy Noether Lecture at the International Congress of Mathematicians. Her lectures were works of art. Without fail they were accessible to nonexperts, told a compelling and creative story, delighted her audiences with literary allusions and puns, and invited everyone into the fun.

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Tom Halverson is a professor of mathematics at Macalester College. His email address is halverson@macalester.edu.

Arun Ram is a professor of mathematics at the University of Melbourne. His email address is aram@unimelb.edu.au.

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Figure 1. Georgia Benkart at the graduation of her PhD student, Dongho Moon, on May 15, 1998.

Georgia's service to the mathematical profession is leg endary. After retiring from teaching in 2006, she conti ued an active research program in which she publishe nearly 40 papers. At the same time, she focused her tention on service to her professional societies. She wa

Sheila Sundaram Arturo Pianzola Ruth Charney Ellen Kirkman Rolf Farnsteiner Alberto Elduque Konstantina Christodoulopoulou Sarah Witherspoon Efim Zelmanov

Rosa Orellana

Alejandro Adem

Hèléne Barcelo

Dan Nakano

An Influential Scholar and Generous

Collaborator

MathSciNet 2023

134 papers

> 1750 citations

3 memoirs

96 coauthors

Allison, Bruce N. Bahturin, Yuri Barnes, Jeffrey M. Biswal, Rekha Britten, Daniel J. Chakrabarti, Manish Viktor Cho, Soojin¹ Colmenarejo Hernando, Laura Diaconis, Persi W. Doty, Stephen R. Elduque, Alberto Eng, Oliver D. Feldvoss, Jörg Fernández López, Antonio¹ Frenkel, Igor Borisovich Futorny, Vyacheslav M. Gaglione, Anthony M. Gao, Yun¹ Gray, Mary Wheat Gregory, Thomas B. Guay, Nicolas Halverson, Thomas Michael Harman, Nate Harris, Pamela E. Isaacs, Irving Martin Joyner, William David Jung, Ji Hye Kang, Seok-Jin Kashiwara, Masaki Kaplansky, Irving Kashuba, Iryna Kass, Kidwell, Mark E. Kirkman, Ellen E. Steven Neil Klivans, Caroline J. Kostrikin, Alekseĭ Ivanovich Kuznetsov, Michael I. Labra, Alicia

Lauter, Kristin E. Leduc, Robert Edgar Lee Shader, Chanyoung Lee, Hyeonmi Lee, Kyu-Hwan Lemire, Frank Liebeck, Martin W. Lopes, Samuel A. Madariaga Merino, Sara Martínez López, Consuelo Maycock, Ellen J. McCrimmon, Kevin Meinel, Joanna Melville, Duncan J. Meyerson, Mark D. Misra, Kailash C. Moody, Robert V. Moon, Dong Ho Neher, Erhard Nguyen, Van Cat Oh, Se-jin Ondrus, Matthew Orellana, Rosa C. Osborn, I. Marshall Panova, Greta Cvetanova Park, Euiyong Pereira, Mariana Premet, Alexander A. Pérez-Izquierdo, José María Ram, Reiner, Victor Roby, Tom Rothschild, Linda Preiss Saltman, Schilling, Anne Seligman, George B. Shin, Dong-Uy Smirnov, Oleg N. Sottile, Frank Spellman, Dennis Srinivasan, Strade, Helmut Stroomer, Jeffrey Terwilliger, Paul M. Bhama Tiep, Pham Huu Townsend, Douglas W. Wardlaw, William Patterson Wiegand, Sylvia M. Wilcox, Stewart Wilson, Robert Lee Witherspoon, Sarah J. Xu, Xiao Ping³ Yip, Martha Yoshii, Yōji Zel'manov, Efim Isaakovich Zhao, Kaiming Zhu, Jieru

An Extraordinary Teacher and Communicator

- MAA Pólya Lecturer
- AWM-AMS Noether Lecturer
- ICM Noether Lecturer
- Many, many, many keynote addresses, seminar lectures, conference presentations
- \sim \approx 10 big lectures per year for 40 years
- > 2,000 puns delivered



Photo: Seoul ICM 2014, Courtesy IMU

An Incredible Advisor





24 students Suren Fernando Steve Kass Wayne Neidhardt Mark Hall Jeff Stroomer Chanyoung Lee Karl Peters Qing Wang Tom Halverson Rob Leduc Oliver Eng Cheryl Grood Dongho Moon Jeff Hildebrand Matt Bloss Manish Chakrabarti Samuel Lopes Michael Lau Shantala Mukherjee Matt Ondrus Konstantina Christodoulopoulou Sara Madariaga

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A GENERALIZATION OF SUBNET WITH SOME RESULTING IMPROVEMENTS
IN MOORE-SMITH CONVERGENCE THEORY

George Benkart and Douglas W. Townsend
Ohio State University

Section 1. Introduction.

This paper is intended to improve the theory of Moore-Smith Convergence by generalizing the definition of subnet. We begin by examining some short-comings of the present Moore-Smith theory of convergence. Given a net S, it is possible to construct in a natural way a filter dependent on S. From this filter a second net T may be constructed. While S may be shown to be a subnet of T, T in general is not a subnet of S, even though S and T generate the same filter (See example 3). Also, given nets S and T defined on the same directed set, T may equal S on all but one element of the directed set and still not be a subnet of S (See example 1). These limitations in the theory illustrate the need for a new definition of subnet.

The new definition will generalize the classical definition of subnet. It will have the advantage of preserving the classical theorems, while eliminating the above disadvantages. It will also yield the following powerful result:

Given nets S and T, and filters Φ_S and Φ_T constructed from them, $\Phi_S \subseteq \Phi_T$ implies T is a subnet of S under the new definition. In addition, this result will provide an easy method for finding a common supernet for nets S and T.

Section 2. Definition and generalization of subnet.

I. Schur-Weyl Duality



GB: "Contrary to what that picture might suggest, this is a story of cooperation."

Schur-Weyl Duality

(I. Schur, 1900, H. Weyl 1939)

$$G = group \quad V = finite dimensional $\mathbb{C}[G]$ -module$$

$$\mathsf{End}_{\mathsf{G}}(\mathsf{V}^{\otimes k}) = \text{centralizer algebra}$$

$$= \mathsf{Hom}_{\mathsf{G}}(\mathsf{V}^{\otimes k}, \mathsf{V}^{\otimes k})$$

"Diagonal Action"
$$\mathbf{g} \cdot (\mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \mathbf{v}_3 \otimes \mathbf{v}_4 \otimes \mathbf{v}_5 \otimes \mathbf{v}_6)$$

$$= \mathbf{g} \mathbf{v}_1 \otimes \mathbf{g} \mathbf{v}_2 \otimes \mathbf{g} \mathbf{v}_3 \otimes \mathbf{g} \mathbf{v}_4 \otimes \mathbf{g} \mathbf{v}_5 \otimes \mathbf{g} \mathbf{v}_6$$

"Tensor Place Permutation"
$$\mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \mathbf{v}_3 \otimes \mathbf{v}_4 \otimes \mathbf{v}_5 \otimes \mathbf{v}_6$$
 $\mathbf{v}_2 \otimes \mathbf{v}_4 \otimes \mathbf{v}_1 \otimes \mathbf{v}_5 \otimes \mathbf{v}_6 \otimes \mathbf{v}_3$ $\mathbf{v}_2 \otimes \mathbf{v}_4 \otimes \mathbf{v}_1 \otimes \mathbf{v}_5 \otimes \mathbf{v}_6 \otimes \mathbf{v}_3$

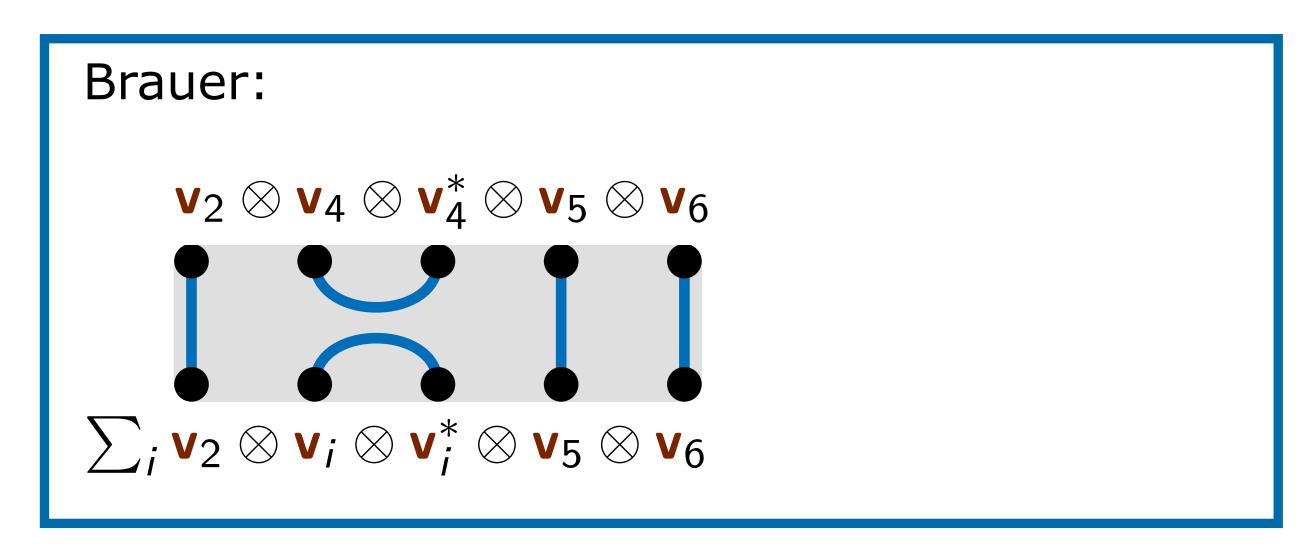
A Tale of Two Groups (AMS-MAA Keynote at JMM 1994, Cincinnati)

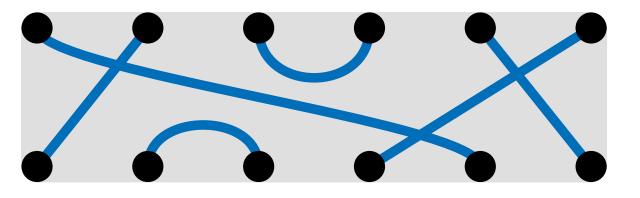
"It was the best of times, it was the worst of times ..."

(I. Schur, 1900)

general linear group =
$$\mathbf{GL}_n$$
 $\mathbf{Z}_k = \mathbb{C}[\mathbf{S}_n]$ symmetric group algebra $n \geq k$

orthogonal group = $\mathbf{O}_n \cdots \mathbf{B}_k(n) = \text{Brauer algebra}$ (R. Brauer, 1937)





(2n-1)!!

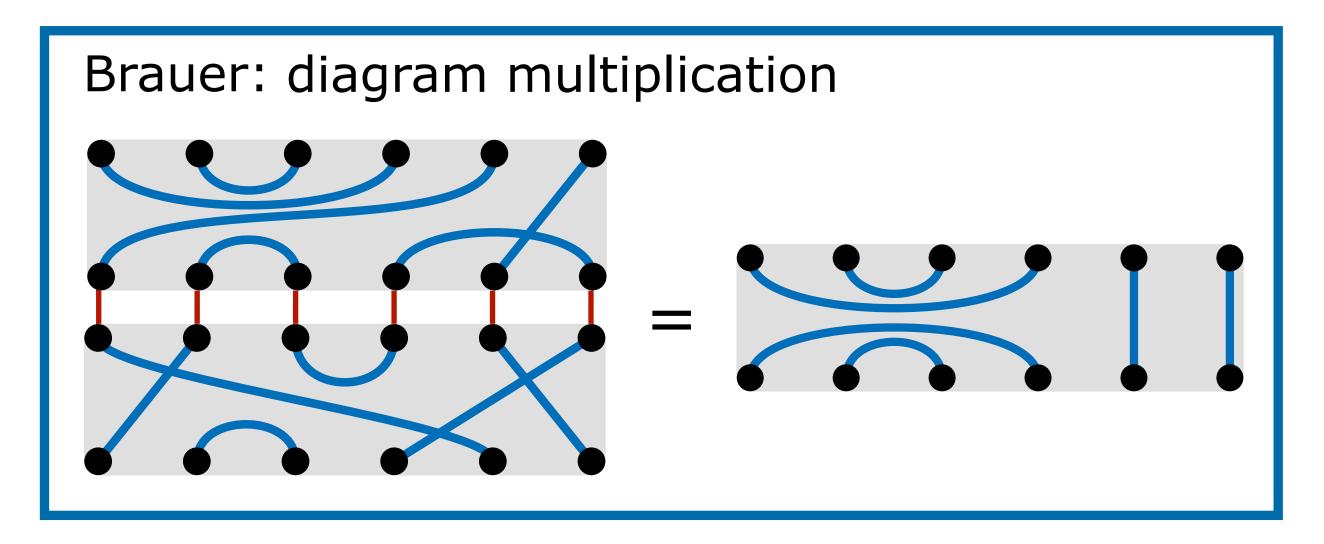
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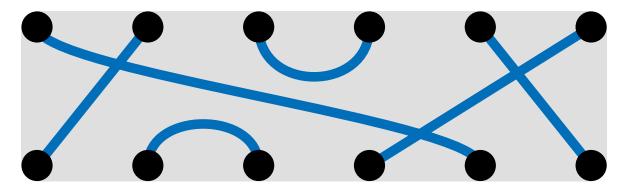
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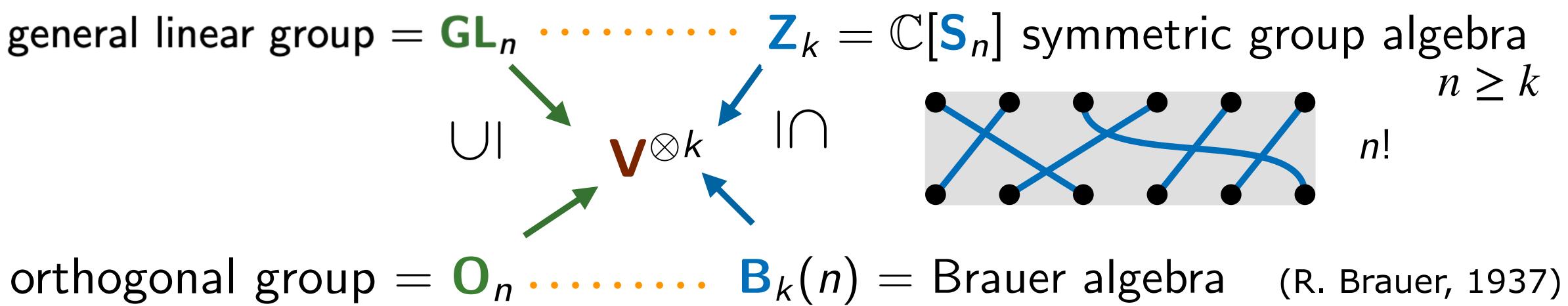


(2n-1)!!

A Tale of Two Groups (AMS-MAA Keynote at JMM 1994, Cincinnati)

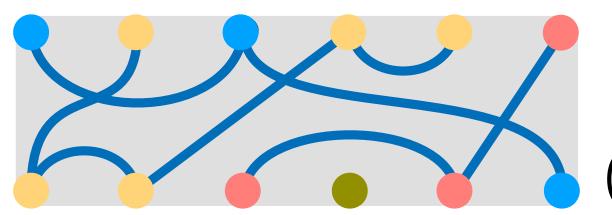
"It was the best of times, it was the worst of times ..."

(I. Schur, 1900)



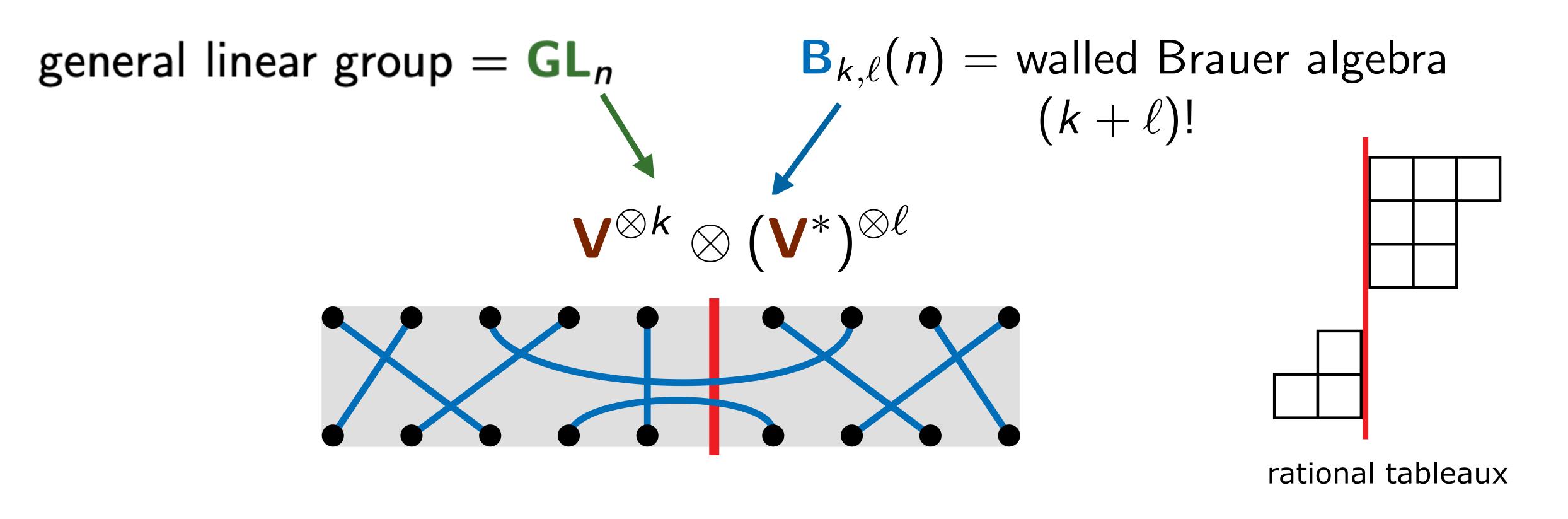
symmetric group =
$$S_n$$
 $P_k(n)$ = partition algebra

P. Martin, 1991-94 VFR Jones, 1993



 B_{2k} Bell #)

Walled Brauer Algebra (1993, JMM 1994: Tale of Two Groups)



BCHLLS: Benkart, Chakrabarti, Halverson, Leduc, Lee, Stroomer

CHILLS: Chakrabarti, Halverson, Leduc, Lee, Stroomer, and I

II. Walking on Graphs



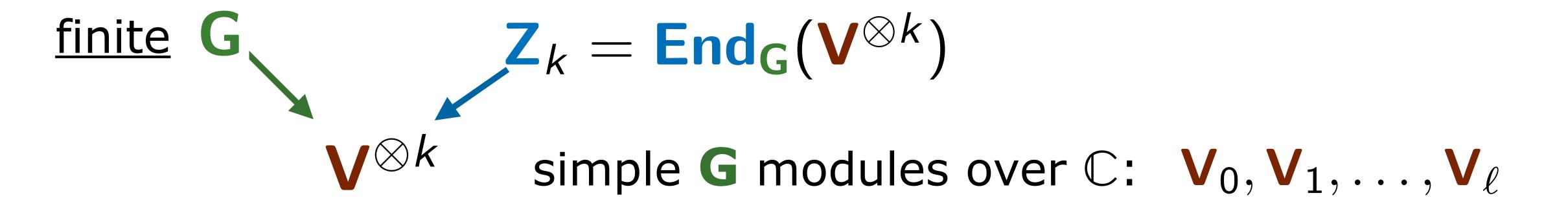
GB: "A pedestrian approach to representation theory"

GB+TH 2012 MAA Mathfest Special Session in Madison, WI: "Walk the Walk, Talk the Talk"

SW Duality: A Story of Cooperation

where $\mathbf{d}_{\lambda} = \dim(\mathbf{G}^{\lambda})$ and $\mathbf{m}_{\lambda} = \dim(\mathbf{Z}_{k}^{\lambda})$

Computing Multiplicities by Walking



$$\mathbf{V}_i \otimes \mathbf{V} = \bigoplus_{j=0}^{\ell} M_{i,j} \mathbf{V}_j$$

 $\mathbf{V}_i \otimes \mathbf{V} = \bigoplus_{j=0}^\ell M_{i,j} \mathbf{V}_j$ $\mathbf{M} = (M_{i,j})$ McKay matrix = adjacency matrix

Representation graph or McKay quiver M_i : directed edges from $M_{i,i}$ directed edges from (i) to (j)

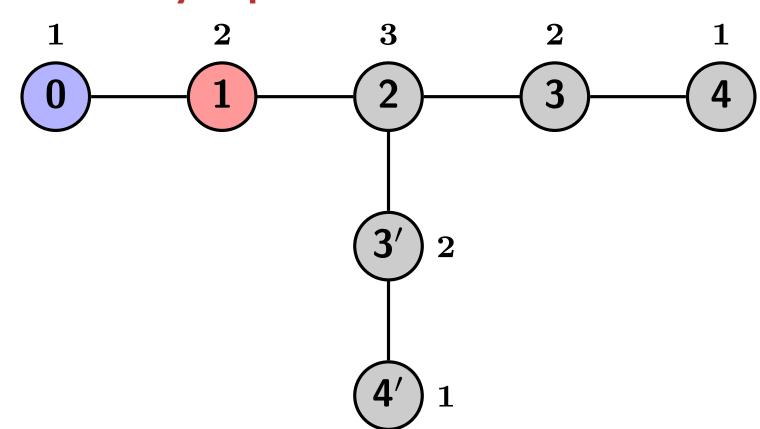
Recursion:

$$\mathbf{V}^{\otimes k} = \mathbf{V}^{\otimes (k-1)} \otimes \mathbf{V} \cong \left(\bigoplus_{i=0}^{\ell} \mathbf{m}_{k-1,i} \mathbf{V}_i \right) \otimes \mathbf{V} = \bigoplus_{i=0}^{\ell} \mathbf{m}_{k-1,i} (\mathbf{V}_i \otimes \mathbf{V})$$

Eg 1: $G = T = 2A_4$ binary tetrahedral group of order 24

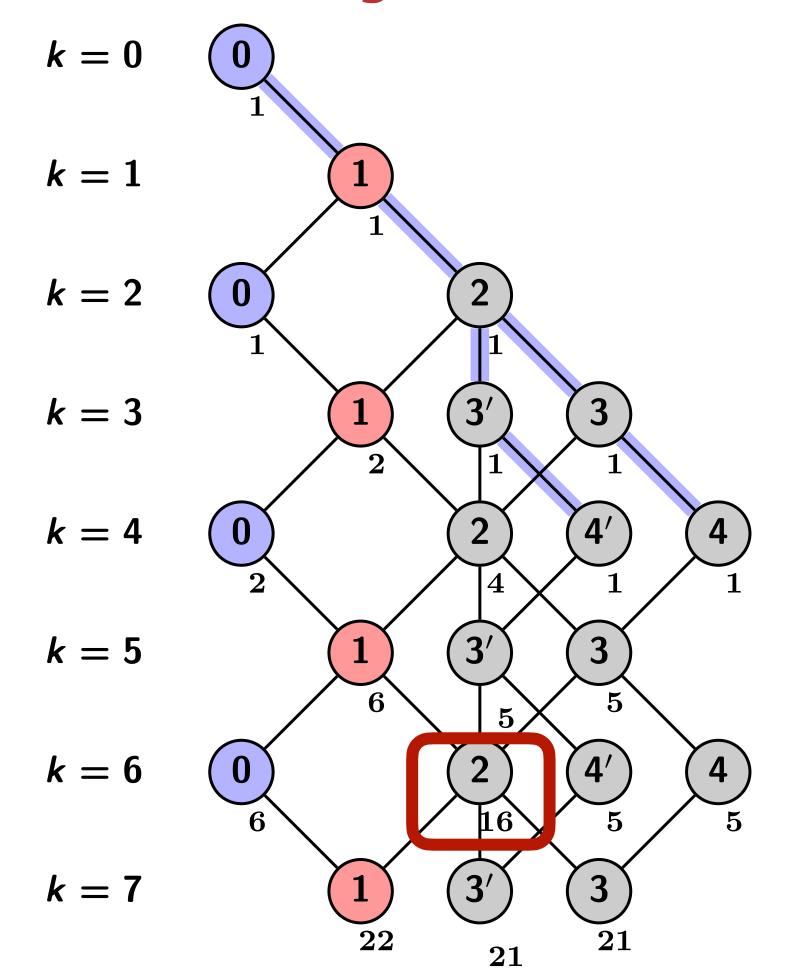
simple modules over \mathbb{C} : $V_0[V_1]V_2, V_3, V_4, V_{3'}, V_{4'}$

McKay quiver



 $\text{mult}_{\mathbf{V} \otimes k}(\mathbf{V}_i) = \# \text{ walks}$ from ① to i) of length k on the McKay quiver

Bratteli diagram



 $V = V_1$ decompose: $V^{\bigotimes k}$

$$\mathsf{mult}_{\mathsf{V}^{\otimes k}}(\mathsf{V}_i)$$

= # paths from 0

to i of length k on
the Bratteli Diagram

$$16 = 6 + 5 + 5$$

$$\operatorname{mult}_{\mathbf{V}^{\otimes k}}(\mathbf{V}_i) = \operatorname{dim}(\mathbf{Z}_k^i)$$

Finite Subgroups of SU(2)

Felix Klein (1900)

John McKay (1980)

1. cyclic group

 \mathbb{Z}_n

2. binary dihedral 4n $\mathbb{D}_n = 2\mathbf{D}_n$

3. binary tetrahedral 24 $\mathbb{T}=2A_4$



4. binary octahedral 48 $\bigcirc = 2S_4$

5. binary icosahedral 120 $\mathbb{I} = 2A_5$

1-1 correspondence with the simply-laced affine Dynkin diagrams of type A-D-E

McKay Centralizer Algebras (Barnes-B-H, 2016)

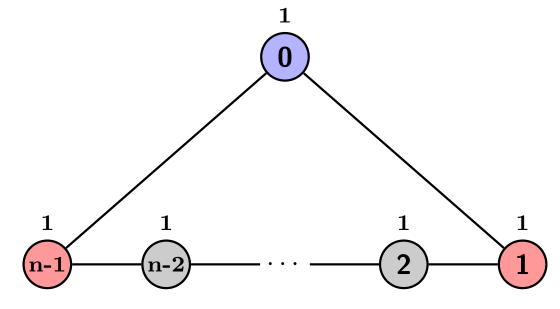
G finite subgroup of SU(2)

$$V = \mathbb{C}^2$$

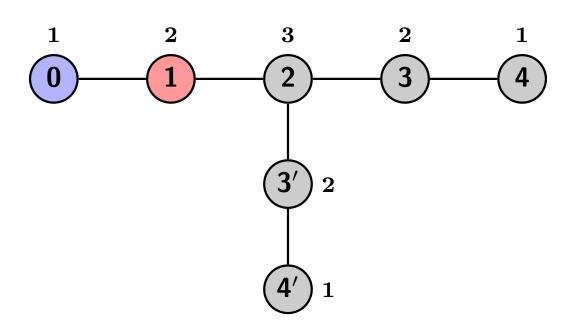
$$Z_k = \operatorname{End}_{\mathbf{G}}(\mathbf{V}^{\otimes k})$$

$$\cong \langle \mathsf{TL}_k, \mathsf{p}_b \mid b \in \mathsf{branch} \rangle$$

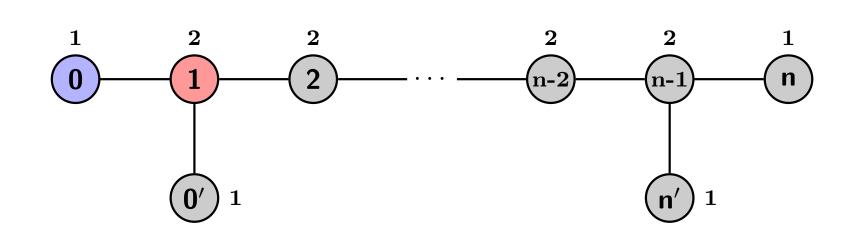
Temperley-Lieb algebra + projector at each branch in the Dynkin diagram $(\mathbf{C}_n, \hat{\mathsf{A}}_{n-1})$



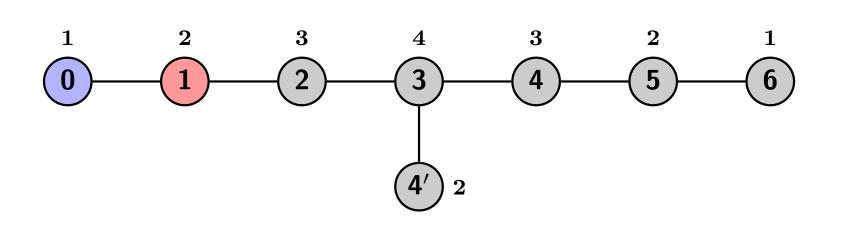
 $(\mathbf{T},\hat{\mathsf{E}}_6)$

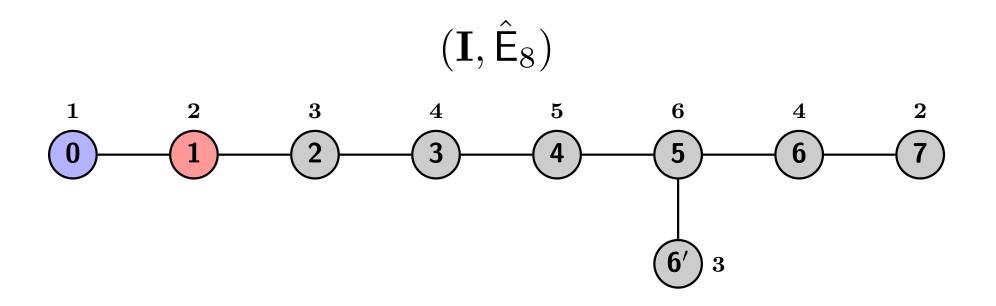


 $(\mathbf{D}_n,\hat{\mathsf{D}}_{n+2})$



 $(\mathbf{O},\hat{\mathsf{E}}_7)$





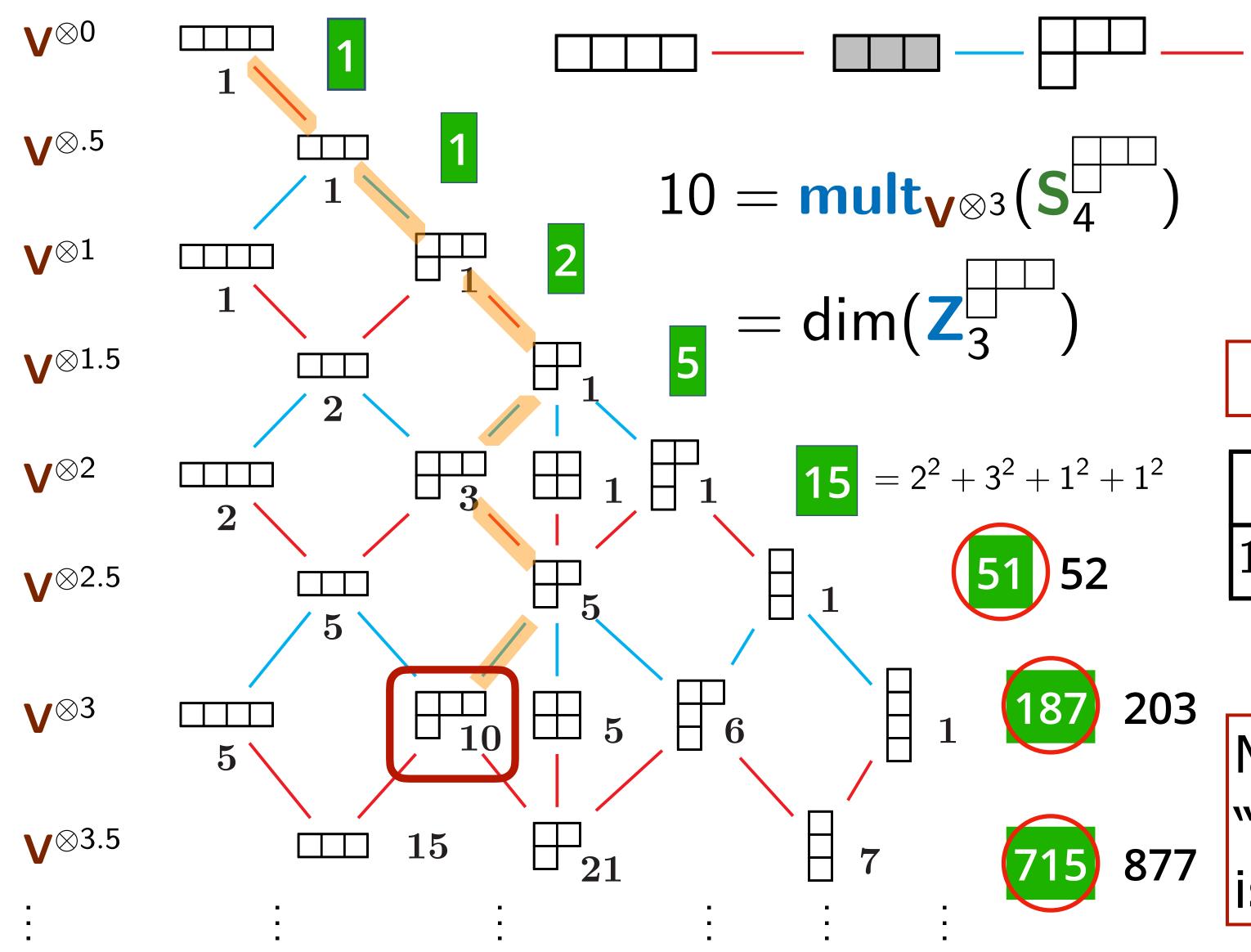
Eg.2: $G = S_n$ V = permutation module

$$\mathbf{V}^{\otimes k}\cong igoplus_{\lambda\vdash n} \mathbf{m}_{k,\lambda} \mathbf{S}^{\lambda}$$
 decomposition as an \mathbf{S}_n module

Tensor Identity: $S^{\lambda} \otimes V \cong Ind_{S_{n-1}}^{S_n} Res_{S_{n-1}}^{S_n}(S^{\lambda})$

$$\cong \bigoplus_{\nu=\mu+\square} \bigoplus_{\mu=\lambda-\square} \mathbf{S}^{\nu}$$

Representation Graph: (S_4, S_3) and $P_k(4)$



Set Partition Tableaux

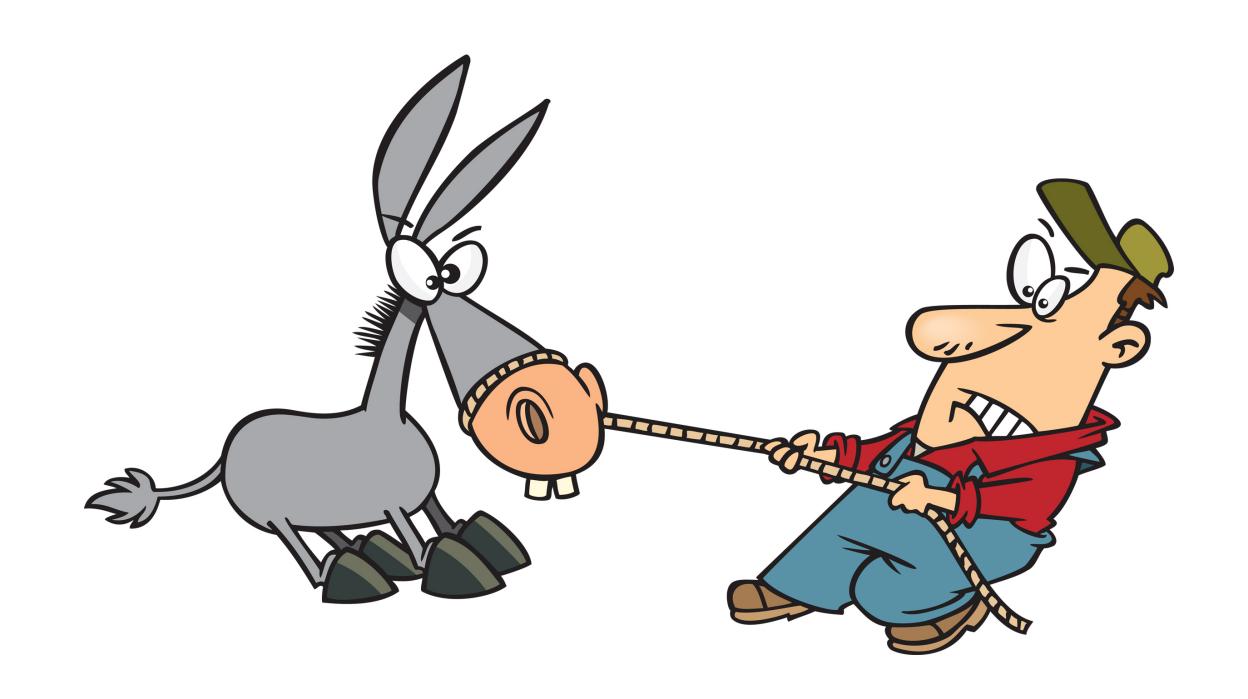
··3[BH]1,2[Orellana, Zabrocki]

[Colmenarejo, Orellana, Saliola, Schilling, Zabrocki]

Martin-Woodcock '99:

"It is possible that the kernel is idempotently generated"

III. Invariant Theory



GB: What's an Invariant? It's something that doesn't move when you act on it

Classical Invariant Theory

- (a) $\Psi: \mathbb{C}[\mathbf{S}_k] \longrightarrow \mathbf{End}_{\mathbf{GL}_n}(\mathbf{V}^{\otimes k})$ [Schur, 1900 Thesis] (FFT) The map is surjective for all k. (SFT) If $k \leq n$ then $\ker(\Psi) = 0$, the only relations come from \mathbf{S}_k If k > n then $\ker(\Psi) = \langle e_{n+1} \rangle$ where $e_{n+1} = \sum_{\sigma \in \mathbf{S}_{n+1}} \operatorname{sign}(\sigma) \sigma$
- (b) $\Psi: \mathbf{B}_k(n) \longrightarrow \mathbf{End}_{\mathbf{O}_n}(\mathbf{V}^{\otimes k})$ [R. Brauer 1937]

 (FFT) The map is surjective for all k

 (SFT) If $k \leq n$ then $\ker(\Psi) = 0$.

 If k > n, then $\ker(\Psi) = \langle e \rangle$ is generate by an idempotent e.

[Hu-Xiao, '10], [Leher-Zhang, '12,'15], [Rubey-Westbury, '14,'15]

Invariant Theory

(c) $\Psi : P_k(n) \longrightarrow \operatorname{End}_{S_n}(V^{\otimes k})$

(FFT) The map is surjective for all k [Martin], [Jones] and is injective if $k \le n/2$.

(SFT) What is the kernel of the Partition algebra?

Invariant Theory

(c)
$$\Psi : \mathbf{P}_k(n) \longrightarrow \mathbf{Ends}_n(\mathbf{V}^{\otimes k})$$

(FFT) The map is surjective for all k [Martin], [Jones] and is injective if $k \le n/2$.

(SFT)
$$\ker(\Psi) = \begin{cases} 0, & k \leq \frac{n}{2} \\ \langle e_{k,n} \rangle, & \frac{n}{2} < k \leq n \\ \langle e_{n,n} \rangle, & k > n \end{cases}$$

[GB-TH] ([Rubey-Westbury])

GB: "essential essential idempotents"

Orbit Basis

So much more...

Extensions, generalizations of these topics

With Dongho Moon

Poincare Series and Invariant Theory via McKay quivers and Characters With Persi Diaconis, Martin Liebeck, Pham Huu Diep

Tensor Product Markov
Chains using the McKay
Correspondence

With Rekha Biswal, Ellen Kirkman, Van Nguyen, Jieru Zhu

McKay Matrices for finite dimensional Hopf algebras

With Laura Colmenarejo, Pamela Harris, Rosa Orellana, Greta Panova, Anne Schilling, Martha Yip

Minimaj-preserving crystals on ordered multiset partitions

With Alberto Elduque

Cross products, invariants, and centralizers

With Caroline Klivans, and Vic Reiner

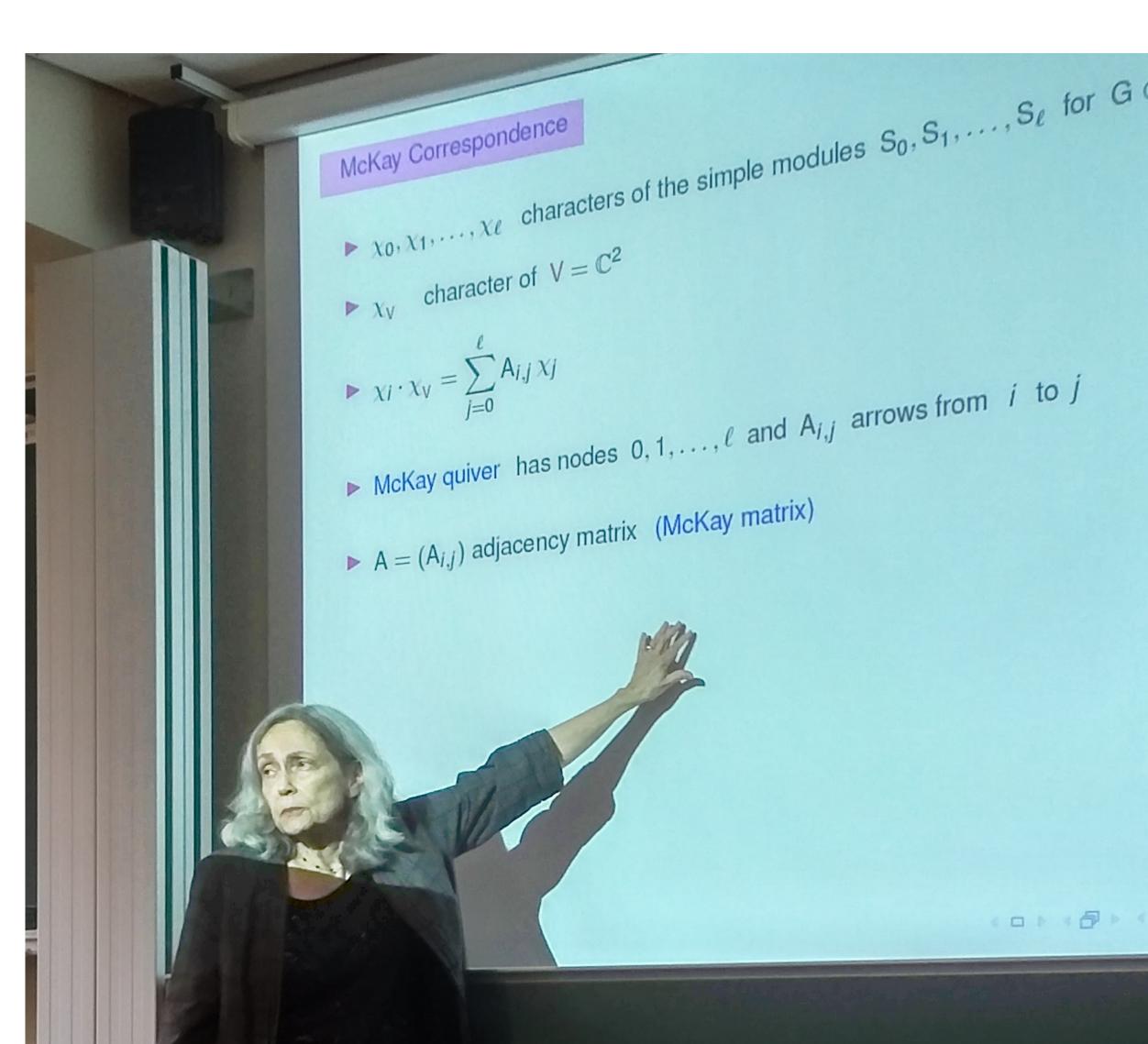
Chip Firing and Critical
Groups using McKay
Quivers

Georgia in "Retirement"

Since 2007 ...

- 46 papers
- 6 big awards
- > 100 lectures!

- AWM President
- MSRI Trustee
- AIM Advisory Panel
- AMS Associate Secretary
 - Council Member
 - JMM Planning(4)
 - 3 prize committees
- Supportive collaborations
- Mentoring



WINART 2016 Women in Noncommutative Algebra and Representation Theory



Algebraic Combinatorixx 2 (BIRS 2017)



WINART 2019 Women in Noncommutative Algebra and Representation Theory



Southeast Lie Theory Seminar

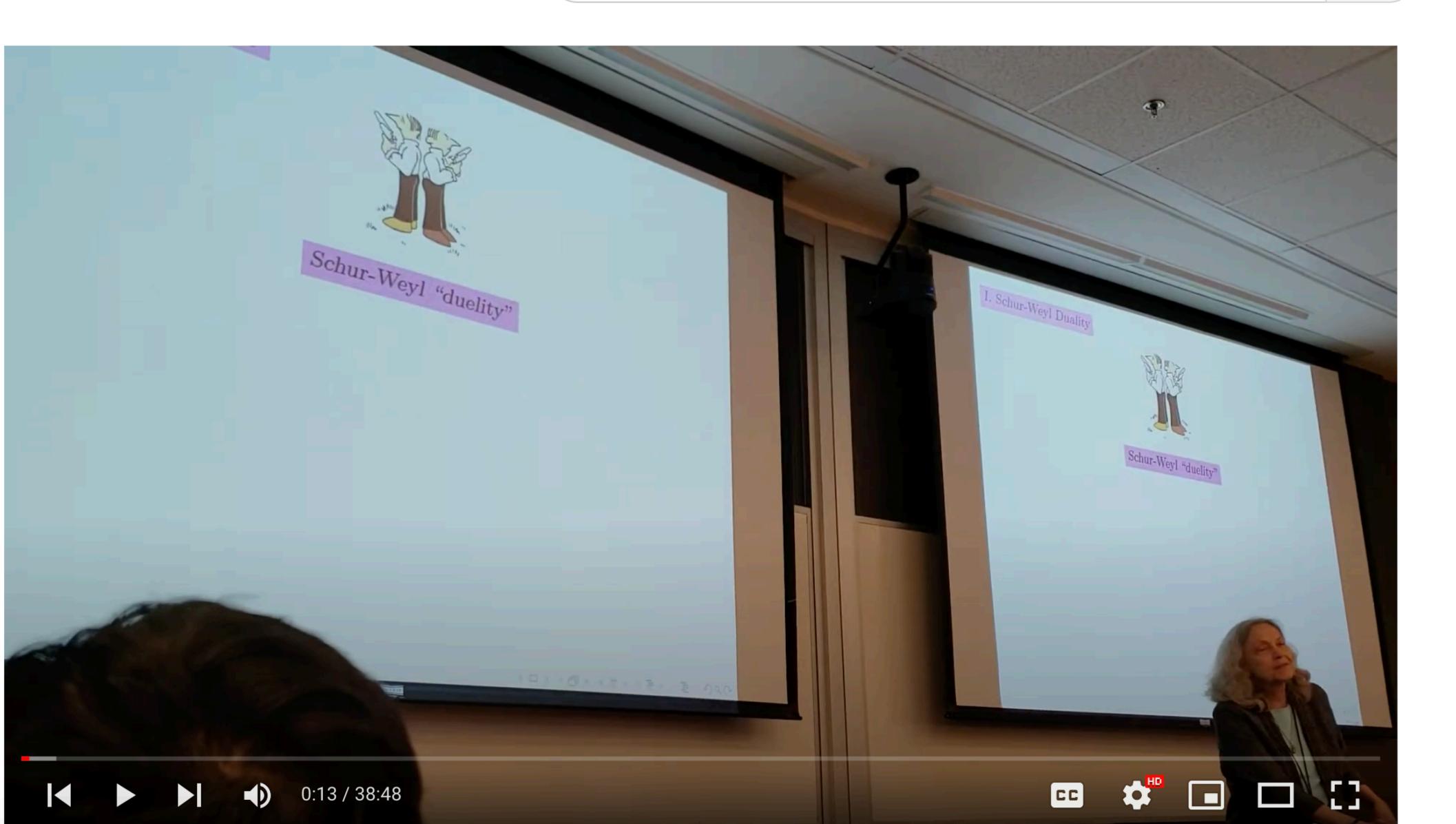


Univ. of Georgia 2018



Search





Georgia Benkart



Her enthusiasm for mathematics was infectious. She provided expert mathematical and professional advice, was incredibly patient as we made mistakes, and wrote countless letters of recommendation.

Even when intensely busy in her role as an AMS associate secretary, Georgia would somehow magically appear at the AMS session when one of our group members spoke.

Georgia was a kind, gentle, warm, brilliant, humble, and generous person, with a wonderful sense of humor and much common sense.

Ellen Kirkman

AMS Notices Memorial Article

March 2023