A Catalanimal formula for Macdonald polynomials

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$R_+ = \{ \alpha_{ij} \mid 1 \leq i < j \leq n \}$ denotes the set of positive roots for $GL_n$, where $\alpha_{ij} = \epsilon_i - \epsilon_j$. 

A root ideal $\Psi \subseteq R_+$ is an upper order ideal of positive roots.
$R_+ = \{ \alpha_{ij} \mid 1 \leq i < j \leq n \}$ denotes the set of positive roots for $GL_n$, where $\alpha_{ij} = \epsilon_i - \epsilon_j$.

A root ideal $\Psi \subseteq R_+$ is an upper order ideal of positive roots.

$\Psi = \text{Roots above Dyck path}$
Let $\Lambda(X)$ be the ring of symmetric functions in $X = x_1, x_2, \ldots$

$h_d = h_d(X) = \sum_{i_1 \leq \ldots \leq i_d} x_{i_1} \cdots x_{i_d}$ with $h_0 = 1$ and $h_d = 0$ for $d < 0$.

For any $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{Z}^n$,

$$s_\gamma = s_\gamma(X) = \det(h_{\gamma_i+j-i}(X))_{1 \leq i,j \leq n}$$
Symmetric functions and Schur functions

- Let $\Lambda(X)$ be the ring of symmetric functions in $X = x_1, x_2, \ldots$
- $h_d = h_d(X) = \sum_{i_1 \leq \ldots \leq i_d} x_{i_1} \cdots x_{i_d}$ with $h_0 = 1$ and $h_d = 0$ for $d < 0$.
- For any $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{Z}^n$,
  \[ s_\gamma = s_\gamma(X) = \det(h_{\gamma_i+j-i}(X))_{1 \leq i, j \leq n} \]

Then,

\[ s_\gamma = \begin{cases} 
\text{sgn}(\gamma + \rho)s_{\text{sort}(\gamma+\rho)-\rho} & \text{if } \gamma + \rho \text{ has distinct nonnegative parts}, \\
0 & \text{otherwise},
\end{cases} \]

- $\text{sort}(\beta)$ = weakly decreasing sequence obtained by sorting $\beta$,
- $\text{sgn}(\beta)$ = sign of the shortest permutation taking $\beta$ to $\text{sort}(\beta)$. 
Weyl symmetrization

Define the Weyl symmetrization operator $\sigma : \mathbb{Q}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}] \to \Lambda(X)$ by linearly extending

$$z^\gamma \mapsto s_\gamma(X)$$

where $z^\gamma = z_1^{\gamma_1} \cdots z_n^{\gamma_n}$.
Modified Macdonald polynomials

The modified Macdonald polynomials $\tilde{H}_\mu = \tilde{H}_\mu(X; q, t)$ are Schur positive symmetric functions in $X = x_1, x_2, \ldots$ over $\mathbb{Q}(q, t)$.

They differ from the integral form Macdonald polynomials by

$$\tilde{H}_\mu(X; q, t) = t^{n(\mu)} J_\mu\left[\frac{X}{1-t^{-1}}; q, t^{-1}\right].$$
The modified Macdonald polynomials $\tilde{H}_\mu = \tilde{H}_\mu(X; q, t)$ are Schur positive symmetric functions in $X = x_1, x_2, \ldots$ over $\mathbb{Q}(q, t)$.

They differ from the integral form Macdonald polynomials by $\tilde{H}_\mu(X; q, t) = t^{n(\mu)} J_\mu\left[\frac{X}{1-t^{-1}}; q, t^{-1}\right]$.

$\tilde{H}_{22} = s_4 + (q + t + qt)s_{31} + (q^2 + t^2)s_{22} + (qt + q^2 t + qt^2)s_{211} + q^2 t^2 s_{1111}$
When \( q = 0 \), the modified Macdonald polynomials reduce to the \textit{modified Hall-Littlewood polynomials} \( \tilde{H}_\mu(X; 0, t) \).

\[
\tilde{H}_{22}(X; 0, t) = s_4 + ts_{31} + t^2 s_{22}
\]
$B_\mu = \text{set of roots above block diagonal matrix with block sizes } \mu_{\ell(\mu)}, \ldots, \mu_1$

$B_{3321} =$
A Catalan function for modified Hall-Littlewoods

\( B_\mu = \) set of roots above block diagonal matrix with block sizes \( \mu_\ell(\mu), \ldots, \mu_1 \)

\[ B_{3321} = \]

**Theorem (Weyman, Shimozono-Weyman)**

\[ \tilde{H}_\mu(X; 0, t) = \omega \sigma \left( \frac{z_1 \cdots z_n}{\prod_{\alpha \in B_\mu} (1 - tz^\alpha)} \right), \]

where \( z^\alpha = z_i/z_j \).
Catalan functions for modified Hall-Littlewoods

\[ R_\mu := \{ \alpha_{ij} \in R_+ \mid \text{south}(b_i) \leq b_j \} . \]

\[ R_{3321} = \]
Catalan functions for modified Hall-Littlewood

\[ b_1 \prec b_2 \prec \cdots \prec b_n \]

\[ R_\mu := \{ \alpha_{ij} \in R_+ \mid \text{south}(b_i) \preceq b_j \} \].

\[ R_{3321} = \]

\[ \tilde{H}_\mu(X; 0, t) = \omega \sigma \left( \frac{z_1 \cdots z_n}{\prod_{\alpha \in B_\mu} (1 - tz^\alpha)} \right), \]

\[ = \omega \sigma \left( \frac{z_1 \cdots z_n}{\prod_{\alpha \in R_\mu} (1 - tz^\alpha)} \right) \]
A Catalanimal formula for $\tilde{H}_\mu(X; q, t)$

\[ R_\mu := \{ \alpha_{ij} \in R_+ \mid \text{south}(b_i) \leq b_j \} , \]
\[ \hat{R}_\mu := \{ \alpha_{ij} \in R_+ \mid \text{south}(b_i) < b_j \} . \]
A Catalanimal formula for $\tilde{H}_\mu(X; q, t)$

$$\tilde{H}_\mu := \{ \alpha_{ij} \in R_+ \mid \text{south}(b_i) \leq b_j \},$$

$$\hat{R}_\mu := \{ \alpha_{ij} \in R_+ \mid \text{south}(b_i) < b_j \}.$$

**Theorem (Blasiak-Haiman-Morse-Pun-S.)**

The modified Macdonald polynomial $\tilde{H}_\mu = \tilde{H}_\mu(X; q, t)$ is given by

$$\tilde{H}_\mu = \omega \sigma \left( z_1 \cdots z_n \prod_{\alpha_{ij} \in R_+ \setminus \hat{R}_\mu} (1 - q^{\text{arm}(b_i)} + 1 t^{-\text{leg}(b_i)} z_i / z_j) \prod_{\alpha \in \hat{R}_\mu} (1 - qtz^\alpha) \prod_{\alpha \in R_+} (1 - qz^\alpha) \prod_{\alpha \in R_+} (1 - tzh^\alpha) \right).$$
Example

partition $\mu = 22211$
Example

\[
\begin{align*}
1 - q^\frac{z_1}{z_2} \\
1 - qt^{-1} \frac{z_2}{z_3} \\
1 - q^2 t^{-2} \frac{z_3}{z_5} & \quad 1 - q \frac{z_4}{z_6} \\
1 - q^2 t^{-3} \frac{z_5}{z_7} & \quad 1 - qt^{-1} \frac{z_6}{z_8}
\end{align*}
\]

numerator factors 1 \(\rightarrow q\)arm+1 \(t\)-\(\text{leg} z_i/z_j\)

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
\hline
1 & q & & & & & & & \\
& & q t^{-1} & & & & & & \\
& & & q^2 t^{-2} & & & & & \\
& & & & q t^{-3} & & & & \\
& & & & & & q t^{-1} & & \\
& & & & & & & & \\
\hline
\end{array}
\]

\(\hat{H}_{22211}\)

- \(R_\mu \setminus \hat{R}_\mu\) (\(t\) factors)
- \(\hat{R}_\mu\) (\(t\) and \(qt\) factors)
\( q = t = 1 \) specialization

\[
\omega \sigma \left( \frac{\prod_{\alpha \in R_{\mu} \setminus \hat{R}_{\mu}} (1 - q^\text{arm}(b_i) + 1 t^{-\text{leg}(b_i)} z_i / z_j) \prod_{\alpha \in \hat{R}_{\mu}} (1 - qtz^\alpha)}{\prod_{\alpha \in R_{\mu}} (1 - qz^\alpha) \prod_{\alpha \in \hat{R}_{\mu}} (1 - tz^\alpha)} \right)
\]

\[
= \omega \sigma \left( \frac{\prod_{\alpha \in R_{\mu} \setminus \hat{R}_{\mu}} (1 - z^\alpha) \prod_{\alpha \in \hat{R}_{\mu}} (1 - z^\alpha)}{\prod_{\alpha \in R_{\mu}} (1 - z^\alpha) \prod_{\alpha \in \hat{R}_{\mu}} (1 - z^\alpha)} \right)
\]

\[
= \omega h_1^n
\]

\[
= e_1^n
\]
\[ q = 0 \text{ specialization} \]

\[
\begin{align*}
\omega \sigma &\left( z_1 \cdots z_n \right)_{\alpha_{ij} \in R_{\mu} \setminus \hat{R}_{\mu}} \prod_{\alpha \in \hat{R}_{\mu}} (1 - q^{\text{arm}(b_i)} + t^{-\text{leg}(b_i)} \frac{z_i}{z_j}) \prod_{\alpha \in \hat{R}_{\mu}} (1 - qt z^\alpha) \\
&\xrightarrow{q=0} \omega \sigma \left( \frac{z_1 \cdots z_n}{\prod_{\alpha \in R_{\mu}} (1 - t z^\alpha)} \right) \\
&= \tilde{H}_{\mu}(X; 0, t)
\end{align*}
\]
Proof of formula for $\tilde{H}_\mu$

Definition

$\nabla$ is the linear operator on symmetric functions satisfying

$\nabla \tilde{H}_\mu = t^{n(\mu)} q^{n(\mu^*)} \tilde{H}_\mu$, where $n(\mu) = \sum_i (i - 1) \mu_i$. 
Proof of formula for $\tilde{\mathcal{H}}_\mu$

**Definition**

$\nabla$ is the linear operator on symmetric functions satisfying

$$\nabla \tilde{\mathcal{H}}_\mu = t^{n(\mu)} q^{n(\mu^*)} \tilde{\mathcal{H}}_\mu,$$

where $n(\mu) = \sum_i (i - 1) \mu_i$.

- Start with the Haglund-Haiman-Loehr formula for $\tilde{\mathcal{H}}_\mu$ as a sum of LLT polynomials $\mathcal{G}_\nu(X; q)$. 


Proof of formula for \( \tilde{H}_\mu \)

**Definition**

\( \nabla \) is the linear operator on symmetric functions satisfying

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where \( n(\mu) = \sum_i (i - 1) \mu_i \).

- Start with the Haglund-Haiman-Loehr formula for \( \tilde{H}_\mu \) as a sum of LLT polynomials \( G_\nu(X; q) \).
- Apply \( \omega \nabla \) to both sides.
Proof of formula for $\tilde{H}_\mu$

**Definition**
\[ \nabla \text{ is the linear operator on symmetric functions satisfying } \nabla \tilde{H}_\mu = t^{n(\mu)} q^{n(\mu^*)} \tilde{H}_\mu, \text{ where } n(\mu) = \sum_i (i - 1) \mu_i. \]

- Start with the Haglund-Haiman-Loehr formula for $\tilde{H}_\mu$ as a sum of LLT polynomials $G_\nu(X; q)$.
- Apply $\omega \nabla$ to both sides.
- Use Catalanimal formula for $\omega \nabla G_\nu(X; q)$ and collect terms.
Let $\nu = (\nu_1, \ldots, \nu_k)$ be a tuple of skew shapes.

$\nu = \left( \begin{array}{c}
\begin{array}{cc}
\cdot & \cdot \\
\cdot & \cdot
\end{array}, \\
\begin{array}{cccc}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot
\end{array}
\end{array} \right)$
Let \( \nu = (\nu_1, \ldots, \nu_k) \) be a tuple of skew shapes.

- The content of a box in row \( y \), column \( x \) is \( x - y \).

\[
\nu = \left( \begin{array}{c}
\begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array}
\end{array} \right)
\]

\[
\begin{array}{ccccccc}
-4 & -3 & -2 & -1 & 0 & 1 \\
-3 & -2 & -1 & 0 & 1 & 2 \\
-2 & -1 & 0 & 1 & 2 & 3 \\
-1 & 0 & 1 & 2 & 3 & 4 \\
0 & 1 & 2 & 3 & 4 & 5
\end{array}
\]
Let $\nu = (\nu_1, \ldots, \nu_k)$ be a tuple of skew shapes.

- The *content* of a box in row $y$, column $x$ is $x - y$.
- *Reading order*: label boxes $b_1, \ldots, b_n$ by scanning each diagonal from southwest to northeast, in order of increasing content.

\[
\nu = \left( \begin{array}{c}
\begin{array}{ccc}
& & \\
& & \\
& & \\
\end{array}
\end{array}, \quad \begin{array}{c}
\begin{array}{cccc}
& & & \\
& & & \\
& & & \\
\end{array}
\end{array}\right)
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- A pair $(a, b) \in \nu$ is attacking if $a$ precedes $b$ in reading order and
  - content$(b) = $ content$(a)$, or
  - content$(b) = $ content$(a) + 1$ and $a \in \nu_i$, $b \in \nu_j$ with $i > j$.

$$
\nu = \begin{pmatrix}
\text{ } & \text{ } & \text{ } & b_3 & b_6 \\
\text{ } & \text{ } & \text{ } & b_5 & b_8 \\
b_1 & b_2 & \text{ } & \text{ } & \text{ } \\
b_4 & b_7 & \text{ } & \text{ } & \text{ }
\end{pmatrix}
$$

Attacking pairs: $(b_2, b_3), (b_3, b_4), (b_4, b_5), (b_4, b_6), (b_5, b_7), (b_6, b_7), (b_7, b_8)$
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- $\nu = (\begin{array}{cc}
  & \\
  & \\
  & \\
\end{array},
\begin{array}{cc}
  & \\
  & \\
  & \\
\end{array})$

Attacking pairs: $(b_2, b_3), (b_3, b_4), (b_4, b_5), (b_4, b_6), (b_5, b_7), (b_6, b_7), (b_7, b_8)$
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$$\boldsymbol{\nu} = \left( \begin{array}{ccc} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 \end{array} \right)$$

Attacking pairs: $(b_2, b_3), (b_3, b_4), (b_4, b_5), (b_4, b_6), (b_5, b_7), (b_6, b_7), (b_7, b_8)$
Let $\nu = (\nu(1), \ldots, \nu(k))$ be a tuple of skew shapes.

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\[
\nu = \left( \begin{array}{c}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\hline
\end{array}
\end{array} \right)
\]

Attacking pairs: $(b_2, b_3), (b_3, b_4), (b_4, b_5), (b_4, b_6), (b_5, b_7), (b_6, b_7), (b_7, b_8)$
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\begin{align*}
\nu & = (\begin{array}{|c|c|}
\hline
& \hspace{1cm} \\
\hline
\end{array}, \hspace{1cm} \begin{array}{|c|c|c|}
\hline
& & \\
\hline
\end{array}) \\
\end{align*}

Attacking pairs: $(b_2, b_3), (b_3, b_4), (b_4, b_5), (b_4, b_6), (b_5, b_7), (b_6, b_7), (b_7, b_8)$
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$\nu = (\begin{array}{ccc}
\text{b_1} & \text{b_2} \\
\text{b_4} & \text{b_7} \\
\end{array}, \begin{array}{ccc}
\text{b_3} & \text{b_6} \\
\text{b_5} & \text{b_8} \\
\end{array})$

Attacking pairs: $(b_2, b_3), (b_3, b_4), (b_4, b_5), (b_4, b_6), (b_5, b_7), (b_6, b_7), (b_7, b_8)$
Let $\nu = (\nu(1), \ldots, \nu(k))$ be a tuple of skew shapes.

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\[
\nu = \left( \begin{array}{cc}
\begin{array}{ccc}
\square & \square & \square \\
\square & \square & \square \\
\end{array} & \begin{array}{ccc}
\square & \square & \square \\
\square & \square & \square \\
\square & \square & \square \\
\end{array}
\end{array} \right)
\]

Attacking pairs: $(b_2, b_3), (b_3, b_4), (b_4, b_5), (b_4, b_6), (b_5, b_7), (b_6, b_7), (b_7, b_8)$
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\[
\nu = \left( \begin{array}{cccc}
\text{\includegraphics[scale=0.5]{shape1}} & \text{\includegraphics[scale=0.5]{shape2}} \\
\end{array} \right)
\]

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LLT Polynomials

- A semistandard tableau on $\nu$ is a map $T: \nu \to \mathbb{Z}_+$ which restricts to a semistandard tableau on each $\nu(i)$.

- An attacking inversion in $T$ is an attacking pair $(a, b)$ such that $T(a) > T(b)$.

The LLT polynomial indexed by a tuple of skew shapes $\nu$ is

$$G_\nu(x; q) = \sum_{T \in SSYT(\nu)} q^{\text{inv}(T)} x^T,$$

where $\text{inv}(T)$ is the number of attacking inversions in $T$ and $x^T = \prod_{a \in \nu} x_{T(a)}$. 

$$T = \begin{array}{cc}
\hline
5 & 6 \\
1 & 1 \\
\hline
2 & 4 \\
\hline
3 & 5 \\
\end{array}$$
LLT Polynomials

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5 & 6 \\
1 & 1 \\
2 & 4 \\
3 & 5 \\
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non-inversion
LLT Polynomials

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- An attacking inversion in \( T \) is an attacking pair \((a, b)\) such that \( T(a) > T(b) \).

The LLT polynomial indexed by a tuple of skew shapes \( \nu \) is

\[
G_{\nu}(x; q) = \sum_{T \in \text{SSYT}(\nu)} q^{\text{inv}(T)} x^T,
\]

where \( \text{inv}(T) \) is the number of attacking inversions in \( T \) and \( x^T = \prod_{a \in \nu} x_{T(a)} \).

\[
T = \begin{array}{ccc}
2 & 4 & \\
3 & 5 & \\
1 & 1 & 5 & 6
\end{array}
\]

inversion
A \textit{semistandard tableau} on $\nu$ is a map $T : \nu \to \mathbb{Z}_+$ which restricts to a semistandard tableau on each $\nu(i)$.

An \textit{attacking inversion} in $T$ is an attacking pair $(a, b)$ such that $T(a) > T(b)$.

The \textit{LLT polynomial} indexed by a tuple of skew shapes $\nu$ is

$$
G_\nu(x; q) = \sum_{T \in \text{SSYT}(\nu)} q^{\text{inv}(T)} x^T,
$$

where $\text{inv}(T)$ is the number of attacking inversions in $T$ and $x^T = \prod_{a \in \nu} x_{T(a)}$.

\begin{center}
\begin{array}{cccc}
1 & 1 & & \\
2 & 4 & & \\
3 & 5 & & \\
\end{array}
\end{center}
LLT Polynomials

- A semistandard tableau on $\nu$ is a map $T: \nu \rightarrow \mathbb{Z}_+$ which restricts to a semistandard tableau on each $\nu(i)$.
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The LLT polynomial indexed by a tuple of skew shapes $\nu$ is

$$G_\nu(x; q) = \sum_{T \in \text{SSYT}(\nu)} q^{\text{inv}(T)} x^T,$$

where $\text{inv}(T)$ is the number of attacking inversions in $T$ and $x^T = \prod_{a \in \nu} x_{T(a)}$.

\[ T = \begin{array}{ccc}
2 & 4 & 5 \\
3 & 5 & \text{non-inversion}
\end{array}\]
LLT Polynomials

- A **semistandard tableau** on \( \nu \) is a map \( T : \nu \rightarrow \mathbb{Z}_+ \) which restricts to a semistandard tableau on each \( \nu(i) \).

- An **attacking inversion** in \( T \) is an attacking pair \( (a, b) \) such that \( T(a) > T(b) \).

The **LLT polynomial** indexed by a tuple of skew shapes \( \nu \) is

\[
G_\nu(x; q) = \sum_{T \in \text{SSYT}(\nu)} q^{\text{inv}(T)} x^T,
\]

where \( \text{inv}(T) \) is the number of attacking inversions in \( T \) and \( x^T = \prod_{a \in \nu} x_{T(a)} \).

\[
T = \begin{array}{ccc}
\text{non-inversion} \\
5 & 6 & \\
1 & 1 & \\
2 & 4 & \\
3 & 5 & \\
\end{array}
\]
A semistandard tableau on \( \nu \) is a map \( T: \nu \rightarrow \mathbb{Z}_+ \) which restricts to a semistandard tableau on each \( \nu(i) \).

An attacking inversion in \( T \) is an attacking pair \((a, b)\) such that \( T(a) > T(b) \).

The LLT polynomial indexed by a tuple of skew shapes \( \nu \) is

\[
G_\nu(x; q) = \sum_{T \in \text{SSYT}(\nu)} q^{\text{inv}(T)} x^T,
\]

where \( \text{inv}(T) \) is the number of attacking inversions in \( T \) and \( x^T = \prod_{a \in \nu} x_{T(a)} \).

\[
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inversion
A semistandard tableau on $\nu$ is a map $T : \nu \rightarrow \mathbb{Z}_+$ which restricts to a semistandard tableau on each $\nu(i)$.

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$$T = \begin{array}{ccc}
5 & 6 & \\
1 & 1 & \\
2 & 4 & \\
3 & 5 & \\
\end{array}$$

$\text{inv}(T) = 4$, $x^T = x_1^2 x_2 x_3 x_4 x_5^2 x_6$
The *Catalanimal* indexed by $R_q, R_t, R_{qt} \subseteq R_+$ and $\lambda \in \mathbb{Z}^n$ is

$$H(R_q, R_t, R_{qt}, \lambda) = \sigma \left( \frac{z^\lambda \prod_{\alpha \in R_{qt}} (1 - qtz^\alpha)}{\prod_{\alpha \in R_q} (1 - qz^\alpha) \prod_{\alpha \in R_t} (1 - tz^\alpha)} \right).$$
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With $n = 3$,

$$H(R_+, R_+, \{\alpha_{13}\}, (111)) = \sigma \left( \frac{z^{111}(1 - qt z_1/z_3)}{\prod_{1 \leq i < j \leq 3}(1 - q z_i/z_j)(1 - t z_i/z_j)} \right)$$

$$= s_{111} + (q + t + q^2 + qt + t^2)s_{21} + (qt + q^3 + q^2 t + qt^2 + t^3)s_3$$

$$= \omega \nabla e_3.$$
For a tuple of skew shapes $\nu$, the LLT Catalanimal $H_{\nu} = H(R_q, R_t, R_{qt}, \lambda)$ is determined by

- $R_+ \supseteq R_q \supseteq R_t \supseteq R_{qt}$,
- $R_+ \setminus R_q =$ pairs of boxes in the same diagonal,
- $R_q \setminus R_t =$ the attacking pairs,
- $R_t \setminus R_{qt} =$ pairs going between adjacent diagonals,
- $\lambda$: fill each diagonal $D$ of $\nu$ with
  \[1 + \chi(D \text{ contains a row start}) - \chi(D \text{ contains a row end}).\]
Listing this filling in reading order gives $\lambda$. 
LLT Catalananimals

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- $R_{qt} = \text{all other pairs,}$

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LLT Catalananimals

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- $\lambda$: fill each diagonal $D$ of $\nu$ with $1 + \chi(D$ contains a row start$) - \chi(D$ contains a row end$).$
Theorem (Blasiak-Haiman-Morse-Pun-S.)

Let \( \nu \) be a tuple of skew shapes and let \( H_\nu = H(R_q, R_t, R_{qt}, \lambda) \) be the associated LLT Catalananimal. Then

\[
\nabla G_\nu(X; q) = c_\nu \omega \text{pol}_X(H_\nu)
\]

\[
= c_\nu \omega \text{pol}_X \sigma \left( \frac{z^\lambda \prod_{\alpha \in R_{qt}} (1 - qt z^\alpha)}{\prod_{\alpha \in R_q} (1 - q z^\alpha) \prod_{\alpha \in R_t} (1 - t z^\alpha)} \right)
\]

for some \( c_\nu \in \pm q^{\mathbb{Z}} t^{\mathbb{Z}} \).
Theorem (Haglund-Haiman-Loehr, 2005)

\[ \tilde{H}_\mu(X; q, t) = \sum_D \left( \prod_{u \in D} q^{-\text{arm}(u)} t^{\text{leg}(u)+1} \right) G_{\nu(\mu, D)}(X; q), \]

where

- the sum runs over all subsets \( D \subseteq \{(i, j) \in \mu \mid j > 1\} \), and
- \( \nu(\mu, D) = (\nu^{(1)}, \ldots, \nu^{(k)}) \) where \( k = \mu_1 \) is the number of columns of \( \mu \), and \( \nu^{(i)} \) is a ribbon of size \( \mu_i^* \), i.e., box contents \( \{-1, -2, \ldots, -\mu_i^*\} \), and descent set \( \text{Des}(\nu^{(i)}) = \{-j \mid (i, j) \in D\} \).
Haglund-Haiman-Loeehr formula example

\[
\begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & 4 \\
\end{array}
\]

\[
D = \{b_1, b_2, b_3\}
\]

\[
q^{-1}t^4
\]

\[
D = \{b_1, b_2, b_3\}
\]

\[
q^{-1}t^3
\]

\[
D = \{b_2, b_3\}
\]

\[
q^{-1}t^3
\]

\[
D = \{b_1, b_2\}
\]

\[
t^2
\]

\[
D = \{b_1, b_3\}
\]

\[
q^{-1}t^2
\]

\[
D = \{b_2\}
\]

\[
t
\]

\[
D = \{b_3\}
\]

\[
q^{-1}t
\]

\[
D = \{b_1\}
\]

\[
t
\]

\[
D = \emptyset
\]
Putting it all together

Take HHL formula \( \tilde{H}_\mu = \sum_D a_{\mu,D} G_{\nu(\mu,D)} \) and apply \( \omega \nabla \).
Putting it all together

- Take HHL formula $\tilde{H}_\mu = \sum_D a_{\mu,D} G_{\nu(\mu,D)}$ and apply $\omega \nabla$.
- By construction, all the LLT Catalanimals $H_{\nu(\mu,D)}$ appearing on the LHS will have the same root ideal data $(R_q, R_t, R_{qt})$. 
Putting it all together

- Take HHL formula $\tilde{H}_\mu = \sum_D a_{\mu,D} G_{\nu(\mu,D)}$ and apply $\omega \nabla$.
- By construction, all the LLT Catalanimals $H_{\nu(\mu,D)}$ appearing on the LHS will have the same root ideal data $(R_q, R_t, R_{qt})$.
- Collect terms to get $\prod_{\alpha_{ij} \in R_\mu \setminus \hat{R}_\mu} (1 - q^{\text{arm}(b_i)} + 1 t^{-\text{leg}(b_i)} z_i / z_j)$ factor.

$$\tilde{H}_\mu = \omega \sigma \left( \frac{\prod_{\alpha_{ij} \in R_\mu \setminus \hat{R}_\mu} (1 - q^{\text{arm}(b_i)} + 1 t^{-\text{leg}(b_i)} z_i / z_j) \prod_{\alpha \in \hat{R}_\mu} (1 - q t z^\alpha)}{\prod_{\alpha \in R_+} (1 - q z^\alpha) \prod_{\alpha \in R_\mu} (1 - t z^\alpha)} \right).$$
A positivity conjecture

What can this formula tell us that other formulas for Macdonald polynomials do not?
A positivity conjecture

What can this formula tell us that other formulas for Macdonald polynomials do not?

\[ \tilde{H}_{\mu}^{(s)} := \omega \sigma \left( (z_1 \cdots z_n)^s \frac{\prod_{\alpha_{ij} \in R_{\mu} \setminus \hat{R}_{\mu}} \left(1 - q^{\text{arm}(b_i)+1} t^{-\text{leg}(b_i)} z_i / z_j \right) \prod_{\alpha \in \hat{R}_{\mu}} (1 - qt z^\alpha) \prod_{\alpha \in R_+} (1 - q z^\alpha) \prod_{\alpha \in R_{\mu}} (1 - t z^\alpha)}{\prod_{\alpha \in R_{\mu}} (1 - t z^\alpha)} \right) \]

Conjecture (Blasiak-Haiman-Morse-Pun-S.)

For any partition \( \mu \) and positive integer \( s \), the symmetric function \( \tilde{H}_{\mu}^{(s)} \) is Schur positive. That is, the coefficients in

\[ \tilde{H}_{\mu}^{(s)} = \sum_{\nu} K_{\nu,\mu}^{(s)}(q, t) s_\nu(X) \]

satisfy \( K_{\nu,\mu}^{(s)}(q, t) \in \mathbb{N}[q, t] \).
Thank you!


