Toward Butler's conjecture

Jaeseong Oh
Korea Institute for Advanced Study

joint w/ Donghyun Kim & Seung Jin Lee

Macdonald polynomials

$$\underset{s_\mu[X]}{\text{Schur}} \xrightarrow{(q,t)\text{-analogue}} \underset{P_\mu[X;q,t]}{\text{Macdonald}} \xrightarrow{\text{scaling}} \underset{J_\mu[X;q,t]}{\text{integral form}} \xrightarrow[t\text{-reversal}]{\text{plethystic substitution}} \underset{\tilde{H}_\mu[X;q,t]}{\text{modified}}$$

<u>Thm</u> (Haiman 01, Macdonald positivity conjecture) For $\mu \vdash n$,

let $\tilde{H}_\mu[X;q,t] = \sum_{\lambda \vdash n} \tilde{K}_{\lambda\mu}(q,t)\, s_\lambda[X]$.

Then $\underline{\tilde{K}_{\lambda\mu}(q,t)} \in \mathbb{Z}_{\geq 0}[q,t]$

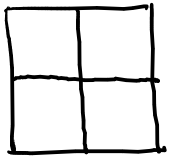
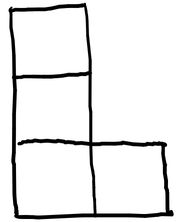
modified (q,t)-Kostka polynomial

<u>Open Problem</u> Find a combinatorial interpretation for $\tilde{K}_{\lambda\mu}(q,t)$

An observation

For $\mu \vdash n$, $T_\mu := \prod_{c \in \mu} q^{\mathrm{coarm}_\mu(c)} \, t^{\mathrm{coleg}_\mu(c)}$

eg.

$\lambda = (2,2) =$  , $\mu = (2,1,1) =$ 

$$T_\lambda = \begin{array}{|c|c|}\hline t & qt \\ \hline 1 & q \\ \hline \end{array} = q^2t^2 \qquad T_\mu = \begin{array}{|c|c}\hline t^2 & \\ \hline t & \\ \hline 1 & \multicolumn{1}{|c|}{q} \\ \hline \end{array} = qt^3$$

$$\tilde{H}_\lambda = 1\, s_4 + (qt + q + t)\, s_{31} + (q^2 + t^2)\, s_{22} + (q^2t + qt^2 + qt)\, s_{211} + q^2t^2\, s_{1111}$$

$$\tilde{H}_\mu = 1\, s_4 + (t^2 + q + t)\, s_{31} + (qt + t^2)\, s_{22} + (qt^2 + t^3 + qt)\, s_{211} + qt^3\, s_{1111}$$

$1 : 1 \qquad q : t = T_\lambda : T_\mu$

Butler's conjecture

In general, one can observe that for $\nu \vdash n+1$ and $\lambda, \mu \subseteq \nu$ s.t. $|\nu/\lambda| = |\nu/\mu| = 1$, we have

$$\tilde{H}_\lambda = A + T_\lambda B, \quad \text{and}$$

$$\tilde{H}_\mu = A + T_\mu B.$$

ω and $q,t \mapsto q^{-1}, t^{-1}$

Conj (Butler 94) For $\nu \vdash n+1$, let $\lambda, \mu \subseteq \nu$ s.t. $|\nu/\lambda| = |\nu/\mu| = 1$.

Then

$$I_{\lambda,\mu} := \frac{T_\mu \tilde{H}_\lambda - T_\lambda \tilde{H}_\mu}{T_\mu - T_\lambda}$$

is Schur positive

Macdonald intersection polynomial

HHL formula

For a filling ω of μ,

$\boxed{b} \cdots < \cdots \boxed{a}$ or $\boxed{a} \cdots > \cdots \boxed{b}$ $\longrightarrow$ inversion pair

$\begin{array}{c}\boxed{a}\\ \vee \\ \boxed{b}\end{array}$ $\longrightarrow$ descent

$$\mathrm{stat}^{HHL}_{\mu}(\omega) := q^{\mathrm{inv}_\mu(\omega)} \prod_{u\in \mathrm{Des}_\mu(\omega)} q^{-\mathrm{arm}_\mu(u)}\, t^{\mathrm{leg}_\mu(u)+1}$$

Thm (Haglund–Haiman–Loehr 05) For $\mu \vdash n$,

$$\tilde{H}_\mu[X;q,t] = \sum_{\omega\in S_n} \mathrm{stat}^{HHL}_{\mu}(\omega)\, F_{\mathrm{iDes}(\omega)}$$

where F is the (Gessel's) fundamental quasisymmetric function

$$F_{n,S}[X] = F_S[X] := \sum_{\substack{1\le b_1\le \cdots \le b_n \\ i\in S \Rightarrow b_i < b_{i+1}}} x_{b_1}\cdots x_{b_n}$$

Example & naive guess

σ	1234	2134	2314	2341	1324	1342	3124	3142	3412	1243	1423	4123	3214	3241	3421	2143	2413	4213	2431	4231	1432	4132	4312	4321
iDes	ϕ	$\{1\}$			$\{2\}$					$\{3\}$			$\{1,2\}$			$\{1,3\}$					$\{2,3\}$			$\{1,2,3\}$
$\mathrm{stat}^{HHL}_{(2,2)}$	1	q	t	qt	q	qt	t	q^2	t^2	q	qt	t	qt	q^2t	qt^2	q^2	t^2	qt	q^2t	qt^2	q^2t	qt	qt^2	q^2t^2

$$\tilde{H}_{(2,2)}[X;q,t] = F_\phi + (q+t+qt)F_{\{1\}} + (q+qt+t+q^2+t^2)F_{\{2\}} + (q+qt+t)F_{\{3\}}$$
$$+ (qt+q^2t+qt^2)F_{\{1,2\}} + (q^2+t^2+qt+q^2t+qt^2)F_{\{1,3\}} + (q^2t+qt+qt^2)F_{\{2,3\}} + q^2t^2F_{\{1,2,3\}}$$

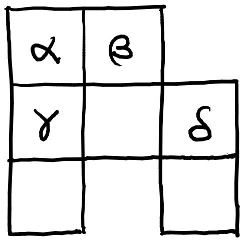
σ	1234	2134	2314	2341	1324	1342	3124	3142	3412	1243	1423	4123	3214	3241	3421	2143	2413	4213	2431	4231	1432	4132	4312	4321
iDes	ϕ	$\{1\}$			$\{2\}$					$\{3\}$			$\{1,2\}$			$\{1,3\}$					$\{2,3\}$			$\{1,2,3\}$
$\mathrm{stat}^{HHL}_{(2,2)}$	1	q	t	qt	q	qt	t	q^2	t^2	q	qt	t	qt	q^2t	qt^2	q^2	t^2	qt	q^2t	qt^2	q^2t	qt	qt^2	q^2t^2
$\mathrm{Stat}^{HHL}_{(2,1,1)}$	1	t	t^2	q	t^2	q	t	qt	t^2	q	t^2	t	t^3	qt	qt^3	qt	t^2	t^3	qt^2	qt	qt^3	qt	t^3	qt^3

$1:1$ $\quad T_\lambda : T_\mu$

Filled diagrams

A filled diagram is a pair (D, f) of a digram D of cells and a function f on $\{\text{cells of } D\} \setminus \{\text{bottom cells of } D\}$. We draw a filled diagram by putting $f(u)$ at each cell u.

eg.



For a filling ω of D, we define $\mathrm{stat}_{(D,f)}$ analogous to stat_μ^{HHL}:

$$\mathrm{stat}_{(D,f)}(\omega) := q^{\mathrm{inv}_D(\omega)} \prod_{u \in \mathrm{Des}_D(\omega)} f(u)$$

(Recall $\mathrm{stat}_\mu^{HHL}(\omega) := q^{\mathrm{inv}_D(\omega)} \prod_{u \in \mathrm{Des}_D(\omega)} q^{-\mathrm{arm}_\mu(u)} t^{\mathrm{leg}_\mu(u)+1}$)

Macdonald polynomials for (D,f)

Def For a filled diagram (D,f) of size n, the (generalized) modified Macdonald polynomial

$$\widetilde{H}_{(D,f)} := \sum_{w \in S_n} \mathrm{stat}_{(D,f)}(w) \cdot F_{\mathrm{iDes}(w)},$$

Lemma cycling rule

$$(D,f) = \begin{array}{ccc} \alpha & & \\ \beta & \gamma & \delta \\ \square & \square & \square \end{array} \qquad\qquad (D',f') = \begin{array}{ccc} & & \alpha \\ & & \beta \\ \gamma & \delta & \square \\ \square & \square & \end{array}$$

$$\Rightarrow \quad \widetilde{H}_{(D,f)} = \widetilde{H}_{(D',f')}$$

Column exchange rule

Prop (Kim–Lee–O. 22+)

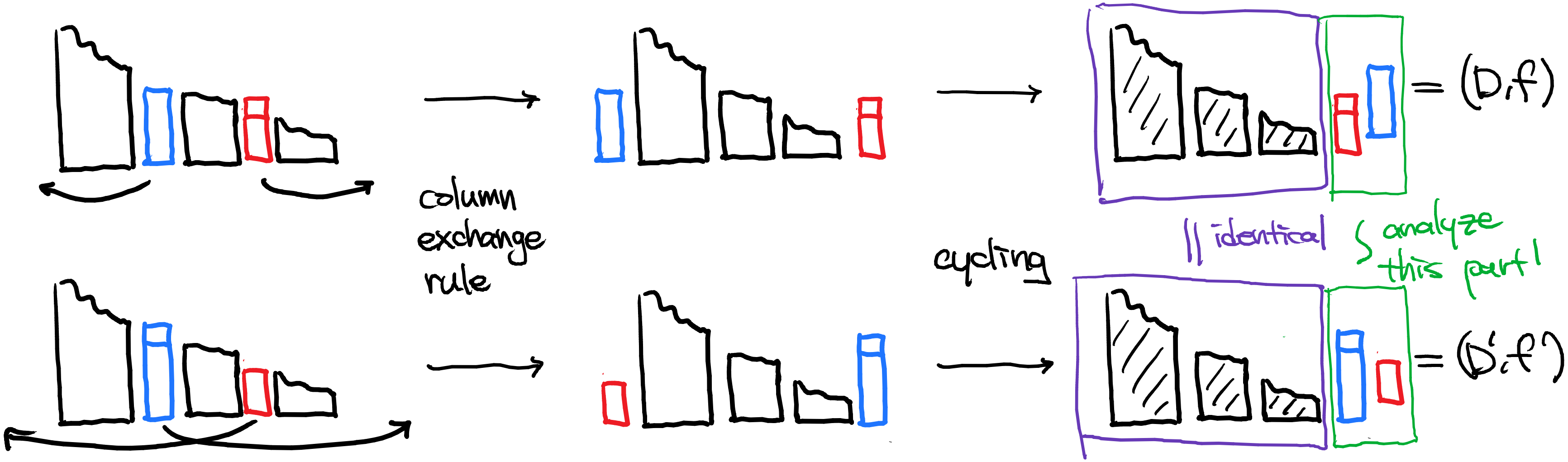
$$(\mu, f_\mu) = \cdots \begin{array}{|c|c}
\hline
b_1 & \\
\hline
\vdots & \\
\hline
b_{n-m-1} & \\
\hline
q\alpha & \\
\hline
\alpha a_1 & \multicolumn{1}{c|}{a_1} \\
\hline
\vdots & \multicolumn{1}{c|}{\vdots} \\
\hline
\alpha a_{m-1} & \multicolumn{1}{c|}{a_{m-1}} \\
\hline
 & \multicolumn{1}{c|}{} \\
\hline
\end{array} \cdots \qquad (\lambda, f_\lambda) = \cdots \begin{array}{c|c|}
\cline{2-2}
 & b_1 \\
\cline{2-2}
 & \vdots \\
\cline{2-2}
 & b_{n-m-1} \\
\cline{2-2}
 & \alpha \\
\hline
\multicolumn{1}{|c|}{a_1} & \alpha a_1 \\
\hline
\multicolumn{1}{|c|}{\vdots} & \vdots \\
\hline
\multicolumn{1}{|c|}{a_{m-1}} & \alpha a_{m-1} \\
\hline
\multicolumn{1}{|c|}{} & \\
\hline
\end{array} \cdots$$

Then there is a stat, iDes, content preserving bijection $\phi : S_N \longrightarrow S_N$. In particular, we have

$$\widetilde{H}_{(\mu, f_\mu)} = \widetilde{H}_{(\lambda, f_\lambda)}$$

Combinatorial formula for $I_{\lambda,\mu}$

In general



We characterize a class $\mathcal{B}$ of permutations and define a bijection δ

s.t. $\begin{cases} \mathrm{stat}_{(D,f)}(w) / \mathrm{stat}_{(D',f')}(\delta(w)) = 1 & \text{if } w \in \mathcal{B} \\ \mathrm{stat}_{(D,f)}(w) / \mathrm{stat}_{(D',f')}(\delta(w)) = T_\lambda / T_\mu & \text{o.w.} \end{cases}$ and $\mathrm{iDes}(w) = \mathrm{iDes}(\delta(w))$

Thm (Kim-Lee-O. 22+)

$$I_{\lambda,\mu} = \sum_{w \in \mathcal{B}_{\lambda,\mu}} \mathrm{stat}_\mu(w)\, F_{\mathrm{iDes}(w)}$$

← Butler permutations

Results on Butler's conjecture

Thm (Kim–Lee–O. 22+) For $\nu \vdash n+1$, let $\lambda, \mu \subseteq \nu$ s.t. $|\nu/\lambda| = |\nu/\mu| = 1$.

- If the cell ν/λ is in the first or second row, then $I_{\lambda,\mu}[X; q, t]$ is Schur positive
- $I_{\lambda,\mu}[X; q, 1]$ is Schur positive

Science Fiction conjecture

<u>Thm (Garsia–Haiman 96, Haiman 01)</u> For $\mu \vdash n$, the Garsia–Haiman module $V_\mu \subseteq \mathbb{C}[x_1, \cdots, x_n, y_1, \cdots, y_n]$ is bigraded S_n-module whose character is the modified Macdonald polynomial:

$$\mathrm{grFrob}(V_\mu; q, t) = \widetilde{H}_\mu[X; q, t]$$

<u>Conj (Bergeron–Garsia 99)</u>

For $\nu \vdash n+1$, and $\mu^{(1)}, \cdots, \mu^{(k)} \subseteq \nu$ s.t. $|\nu/\mu^{(1)}| = \cdots = |\nu/\mu^{(k)}| = 1$.

- $\mathrm{grFrob}\left(\bigcap_{i=1}^{k} V_{\mu^{(i)}}; q, t\right) = \underbrace{I_{\mu^{(1)}, \cdots, \mu^{(k)}}}_{\text{Macdonald intersection polynomial}} := \sum_{i=1}^{k} \prod_{j \neq i} \frac{T_{\mu^{(j)}}}{T_{\mu^{(j)}} - T_{\mu^{(i)}}} \widetilde{H}_{\mu^{(i)}}$

- $\dim\left(\bigcap_{i=1}^{k} V_{\mu^{(i)}}\right) = \frac{n!}{k}$ called $\frac{n!}{k}$-conjecture

Related results

Thm (Kim–Lee–O. 23+)

(a) $e_N^{\perp} I_{\mu^{(1)},\dots,\mu^{(k)}} = 0$

(b) $e_{n-k+1}^{\perp} I_{\mu^{(1)},\dots,\mu^{(k)}} = T_{\bigcap_{i=1}^{k}\mu^{(i)}} \nabla e_{k-1}$

(c) $e_{n-k+1}^{\perp} I_{\mu^{(1)},\dots,\mu^{(k)}} = T_{\bigcap_{i=1}^{k}\mu^{(i)}} D_{k-1}[X;q,t]$

Shuffle theorem [CM18]

Thm (Kim–Lee–O. 23+)

$$I_{\mu^{(1)},\dots,\mu^{(k)}}[X;1,1] = \sum_{\lambda\vdash k-1} (-1)^{k-1-\ell(\mu)} \mathrm{Krew}(\lambda+(1^{\ell(\lambda)}))\, h_{\lambda+(1^{n-k-1})}$$

k	$I_{\mu^{(1)}\dots\mu^{(k)}}[X;1,1]$
1	$h_{11\cdots1}$
2	$h_{21\cdots1}$
3	$2h_{221\cdots1} - h_{311\cdots1}$
4	$5h_{2221\cdots1} - 5h_{321\cdots1} + h_{41\cdots1}$
5	$14h_{22221\cdots1} - 21h_{3221\cdots1} + 3h_{331\cdots1} + 6h_{421\cdots1} - h_{51\cdots1}$

Thank You

[BG99] F. Bergeron, A. Garsia, Science fiction and Macdonald's polynomial, CRM Proceedings & lecture note

[But94] L. Butler, Subgroup lattices and symmetric functions, Mem. AMS

[CM18] E. Carlsson, A. Mellit, A proof of shuffle theorem, JAMS

[GH96] A. Garsia, M. Haiman, Some natural bigraded S_n-modules and q,t-Kostka coefficients, Elec JC

[Hai01] M. Haiman, Hilbert schemes, polygraphs and the Macdonald positivity conjecture, JAMS

[HHL05] J. Haglund, M. Haiman, N. Loehr, A combinatorial formula for Macdonald polynomials, JAMS

[KLO22+] D. Kim, S. J. Lee, J. Oh, Towards Butler's conjecture, preprint, arxiv: 2212.09419

[KLO23+] D. Kim, S. J. Lee, J. Oh, Shuffle formula in science fiction for Macdonald polynomials preprint, arxiv: 2306.14371