Semistandard Parking Functions and a Finite Shuffle Theorem

joint with José Simental and Monica Vazirani

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Goal

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Higher Rank Catalan Polynomials
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A Finite Shuffle Theorem
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**Definition**

An $(m, n)$-dyck path is a lattice path in an $n \times m$ grid from $(n, 0)$ to $(0, m)$ that stays below the diagonal.
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It is well known that \( |(m, n)\text{-Dyck paths}| = C_{(m,n)} \) - the rational Catalan number. The **area** of a dyck path \( D \) = number of boxes between the path and diagonal. The **coarea** = \( \frac{(m-1)(n-1)}{2} \) - area.
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**Definition**

An \((m, n)\)-parking function is a pair \((D, \psi)\) where:
- \(D\) is an \((m, n)\)-dyck path
- \(\psi: \text{north steps of } D \rightarrow \{1, \ldots, m\}\) is a bijection that is increasing on vertical runs (top to bottom)
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- \(\phi: \) north steps of \(D \rightarrow \{1, \ldots, r\}\) is a map that is increasing on vertical runs (top to bottom)

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The **area** of a \((D, \phi)\) = \(\text{area}(D)\)

The **weight** of \((D, \phi)\) = \((|\phi^{-1}(1)|, |\phi^{-1}(2)|, \ldots, |\phi^{-1}(r)|)\).

The **dinv** is...?? We need more.
Affine Compositions and Dinv

The **affine symmetric group** is

\[ \tilde{S}_m := \left\{ \sigma : \mathbb{Z} \rightarrow \mathbb{Z} \mid \sigma \text{ is a bijection, } \sigma(x + m) = \sigma(x) + m, \text{ and } \sum_{i=1}^{m} \sigma(i) = \binom{m+1}{2} \right\}. \]
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A permutation $\sigma \in \tilde{S}_m$ is **n-stable** if $\sigma(x + n) \geq \sigma(x)$ for all $x \in \mathbb{Z}$. Let $\tilde{S}_m^n$ the set of $n$-stable affine permutations.
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**Theorem (Gorsky-Mazin-Vazirani)**

*There is an explicit bijection \( A : PF_{(m,n)} \to \tilde{S}_m^n \).*
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**Theorem (Gorsky-Mazin-Vazirani)**

There is an explicit bijection \( \mathcal{A} : PF_{(m,n)} \to \tilde{S}_m^n \).

Given \( (D, \psi) \in PF_{(m,n)} \) with \( \sigma = \mathcal{A}(D, \psi) \), they defined:

\[
\text{co-dinv}(D, \psi) := |\{(i, h) \in \{1, \ldots, m\} \times \{1, \ldots, n\} \mid \sigma(i + h) < \sigma(i)\}|
\]

So that \( \text{dinv} := \frac{(m-1)(n-1)}{2} - \text{co-dinv} \).
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**Definition**

An \((m, r)\)-affine composition is a function \(f : \mathbb{Z} \to \mathbb{Z}\) such that:

1. \(f(x + m) = f(x) + r\) for all \(x \in \mathbb{Z}\).
2. The set \(f^{-1}\{1, \ldots, r\}\) has exactly one element from each residue class mod \(m\).
3. \(\sum_{x \in f^{-1}\{1, \ldots, r\}} x = \binom{m+1}{2}\).
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**Theorem (González-Simental-Vazirani)**

There is an explicit weight preserving bijection \(A_w : SSPF_r^{(m,n)} \rightarrow n\text{-stable (}m, r\text{)-affine compositions.}
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Gorsky-Mazin-Vazirani proved the affine Springer fiber in the full affine flag variety has an affine paving indexed by parking functions.

We extended this result and showed that the affine spaces in an affine paving of a parabolic affine Springer fiber are indexed by the set \(\text{SSPF}^r_{(m,n)}\), with \(\text{co-dinv}\) measuring the dimension of the corresponding affine space.
To any semistandard parking function we define its \textbf{standardization} via the map \( \text{std} : \text{SSPF}_{(m,n)}^r \rightarrow \text{PF}_{(m,n)} \) defined by

\[
\text{std}(D, \phi) := A^{-1} S^{-1} w(A, D, \phi),
\]

where \( S_w \) is a specific map from certain minimal length coset representatives in \( \tilde{S}_m \) to \((m, r)\)-affine compositions. Define \( \text{dinv} \) by:

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**Define the Anderson function** by \( \gamma : \mathbb{Z}_2 \rightarrow \mathbb{Z}_2 \) by \( \gamma(x, y) = mn - mx - ny \).

**Recipe:** Given a rank \( r \) semistandard parking function \((D, \phi)\) construct the standard parking function \( \text{std}(D, \phi) \) by reading the 1’s, then 2’s, \ldots, then \( r \)'s in order, and then breaking ties using the Anderson labels.
Higher Rank Catalan Polynomials

Definition

The rank $r$ rational $(q, t)$-Catalan polynomials are defined as:

$$C^{(r)}_{(m,n)}(x_1, \ldots, x_r; q, t) := \sum_{(D, \phi) \in SSPF^r(m,n)} q^{\text{area}(D)} t^{\text{dinv}(D, \phi)} x^{\text{wt}(\phi)}.$$
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Define the **Hikita polynomial**

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H_{(m,n)}(X; q, t) := \sum_{(D, \varphi) \in PF(m,n)} q^{\text{area}(D, \varphi)} t^{\text{dinv}(D, \varphi)} Q_{\text{Des}(\sigma^{-1})}(X)
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The higher rank Catalan polynomials interpolate between the rational Catalan numbers and the Hikita polynomial.

$$C^{(1)}_{(m,n)}(x_1) = x_1^m C_{(m,n)}.$$
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$$\mathcal{H}_{(m,n)}(X; q, t) := \sum_{(D,\varphi) \in PF(m,n)} q^{\text{area}(D,\varphi)} t^{\text{dinv}(D,\varphi)} Q_{\text{Des}(\sigma^{-1})}(X).$$

The higher rank Catalan polynomials interpolate between the rational Catalan numbers and the Hikita polynomial.

$$C_{(m,n)}^{(1)}(x_1) = x_1^m C_{(m,n)}.$$ 

$$\lim_{r \to \infty} C_{(m,n)}^{(r)}(x_1, \ldots, x_r; q, t) = \mathcal{H}_{(m,n)}(X; q, t).$$

Thus, $C_{(m,n)}^{(r)}(x_1, \ldots, x_r; q, t)$ are Schur positive, $q, t$-symmetric, and $x_1, \ldots, x_r$-symmetric.
The Rational Shuffle Theorem

Define the **elliptic Hall algebra** $\mathcal{E}$ as the $\mathbb{C}(q, t)$-algebra generated by
$$\{P_{m,n} \mid m, n \in \mathbb{Z}_{\geq 0}\} \text{ (mod relations)}.$$  

Schiffman-Vasserot gave a *geometric* action of $\mathcal{E}$ on symmetric functions $Sym := Q(q, t)[x_1, \ldots]^{S_{\infty}}$. 
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Studying the Khovanov-Rozansky homology of $(m, n)$-torus links, Gorsky-Neguț conjectured:

**Rational Shuffle Theorem (Mellit)**

$$P_{m, n} \cdot (1) = \mathcal{H}_{(m, n)}(X; q, t).$$

When $m = n + 1$ this recovers the classical shuffle theorem.
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When $m = n + 1$ this recovers the classical shuffle theorem.

Define the **spherical DAHA**, $\mathcal{SH}(r)$, as the spherical subalgebra of the DAHA generated by $\{ P_{m,n}^{(r)} \mid m, n \in \mathbb{Z}_{\geq 0} \}$ (mod relations).
A Finite Shuffle Theorem

Theorem (Shiffmann-Vasserot)

The elliptic Hall algebra arises under the inverse limit: \( \mathcal{E} \cong \lim_{\leftarrow} \mathcal{SH}(r) \) where

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Denote by \( \text{Sym}_r := Q(q, t)[x_1, \ldots, x_r]^S_r \) the ring of symmetric polynomials, recall that:
\[ \text{Sym} = \lim_{\leftarrow r} \text{Sym}_r. \]

Hence, the geometric action of \( \mathcal{E} \) on \( \text{Sym} \) induces a geometric action of \( \mathcal{SH}(r) \) on \( \text{Sym}_r. \)
## A Finite Shuffle Theorem

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Hence, the geometric action of \( \mathcal{E} \) on \( \text{Sym} \) induces a geometric action of \( \mathcal{SH}(r) \) on \( \text{Sym}_r \).

Since \( \mathcal{H}_{(m,n)}(X; q, t) = \lim_{\to} C_{(m,n)}^{(r)}(X_r; q, t) \), this yields a finite Shuffle theorem:

\[
P_{(m,n)}^{(r)} \cdot (1) = C_{(m,n)}^{(r)}(x_1, \ldots, x_r; q, t).
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There is a classical *polynomial representation* of $\text{SH}(r)$ on $\text{Sym}_r$.

This induces an *algebraic* action of $\mathcal{E}$ on $\text{Sym}$ that is isomorphic to the geometric representation of Shiffmann-Vasserot.
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$$P_\lambda(X; q, t) \leftrightarrow \tilde{H}_\lambda(X; q, t)$$
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**Theorem (González-Simental-Vazirani)**

The polynomial and geometric representations of $\mathsf{SH}(r)$ are nontrivially isomorphic (they are related via a ‘truncated’ plethysm).
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**Theorem (González-Simental-Vazirani)***

The polynomial and geometric representations of $\mathcal{SH}(r)$ are nontrivially isomorphic (they are related via a ‘truncated’ plethysm).

However, describing the action directly is really hard even at $r = 1$. Maybe next time we meet I’ll know the answer :) .
References

Burban, I., and Schiffmann, O.
On the Hall algebra of an elliptic curve, I.

Gonzalez, N., Simental, J., and Vazirani, M.
Higher Rank \((q, t)\)-Catalan Polynomials, Affine Springer Fibers, and a Finite Rational Shuffle Theorem.

Gorsky, E., Mazin, M., and Vazirani, M.
Affine permutations and rational slope parking functions.

Mellit, A.
Toric braids and \((m, n)\)-parking functions.

THANK YOU!