Semistandard Parking Functions and a Finite Shuffle Theorem

joint with José Simental and Monica Vazirani

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Goal

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Higher Rank Catalan Polynomials

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A Finite Shuffle Theorem

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It is well known that $|(m, n)$-Dyck paths$| = C_{(m,n)}$ - the rational Catalan number.
The **area** of a dyck path D = number of boxes between the path and diagonal.
The **coarea** $= \frac{(m-1)(n-1)}{2}$ - $\texttt{area}$.

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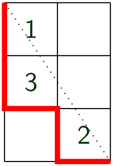

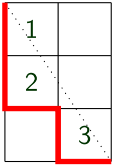

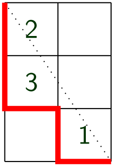

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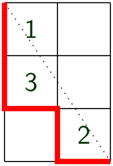

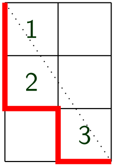

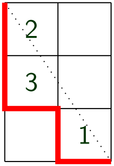

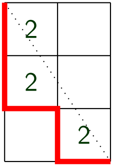

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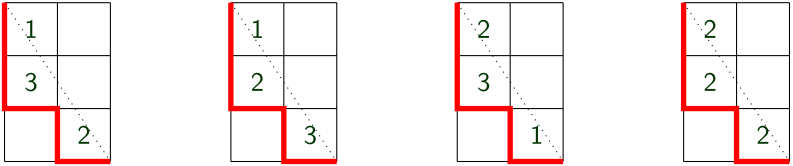

The **area** of a $(\mathrm{D}, \phi) = \mathtt{area}(\mathrm{D})$
The **weight** of $(\mathrm{D}, \phi) = (|\phi^{-1}(1)|, |\phi^{-1}(2)|, \ldots, |\phi^{-1}(r)|)$.
The **dinv** is...?? We need more.

Affine Compositions and Dinv

The **affine symmetric group** is

$$\widetilde{S}_m := \left\{ \sigma : \mathbb{Z} \to \mathbb{Z} \mid \sigma \text{ is a bijection, } \sigma(x+m) = \sigma(x) + m, \text{ and } \sum_{i=1}^{m} \sigma(i) = \binom{m+1}{2} \right\}.$$

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A permutation $\sigma \in \widetilde{S}_m$ is **n-stable** if $\sigma(x+n) \geq \sigma(x)$ for all $x \in \mathbb{Z}$. Let $\widetilde{S}_m^n$ the set of n-stable affine permutations.

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Given $(\mathrm{D}, \psi) \in \mathrm{PF}_{(m,n)}$ with $\sigma = \mathcal{A}(\mathrm{D}, \psi)$, they defined:

$$\texttt{co-dinv}(\mathrm{D}, \psi) := |\{(i,h) \in \{1, \dots, m\} \times \{1, \dots, n\} \mid \sigma(i+h) < \sigma(i)\}|$$

So that $\texttt{dinv} := \frac{(m-1)(n-1)}{2}$ - $\texttt{co-dinv}$.

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Definition

An (m, r)-**affine composition** is a function $f : \mathbb{Z} \to \mathbb{Z}$ such that:

(1) $f(x + m) = f(x) + r$ for all $x \in \mathbb{Z}$.

(2) The set $f^{-1}\{1, \ldots, r\}$ has exactly one element from each residue class mod m.

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There is an explicit weight preserving bijection $\mathcal{A}_w : \mathrm{SSPF}^r_{(m,n)} \to$ *n-stable* (m, r)*-affine compositions.*

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We extended this result and showed that the affine spaces in an affine paving of a parabolic affine Springer fiber are indexed by the set $\mathrm{SSPF}^r_{(m,n)}$, with `co-dinv` measuring the dimension of the corresponding affine space.

To any semistandard parking function we define its **standardization** via the map $\mathsf{std} : \mathsf{SSPF}^r_{(m,n)} \to \mathsf{PF}_{(m,n)}$ defined by

$$\mathsf{std}(\mathsf{D}, \phi) := \mathcal{A}^{-1}\mathcal{S}_{\mathsf{w}}^{-1}\mathcal{A}_{\mathsf{w}}(\mathsf{D}, \phi),$$

where $\mathcal{S}_{\mathsf{w}}$ is a specific map from certain minimal length coset representatives in $\widetilde{S}_m$ to (m, r)-affine compositions. Define **dinv** by:

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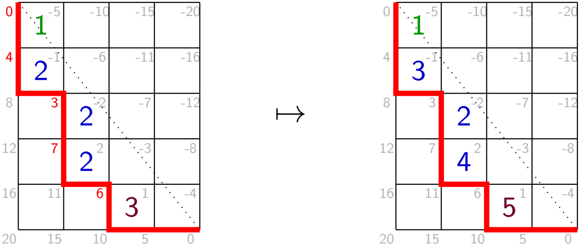
Define the **Anderson function** by $\gamma : \mathbb{Z}_2 \to \mathbb{Z}_2$ by $\gamma(x,y) = mn - mx - ny$.

Recipe: Given a rank r semistandard parking function (D,ϕ) construct the *standard* parking function $\mathrm{std}(\mathrm{D},\phi)$ by reading the 1's, then 2's, . . . , then r's in order, and then breaking ties using the Anderson labels.



Higher Rank Catalan Polynomials

Definition

The **rank r rational (q,t)-Catalan polynomials** are defined as:

$$C^{(r)}_{(m,n)}(x_1,\ldots,x_r;q,t) := \sum_{(\mathrm{D},\phi)\in \mathrm{SSPF}^r(m,n)} q^{\mathtt{area}(\mathrm{D})} t^{\mathtt{dinv}(\mathrm{D},\phi)} x^{\mathtt{wt}(\phi)}.$$

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Define the **Hikita polynomial**

$$\mathcal{H}_{(m,n)}(X;q,t) := \sum_{(\mathrm{D},\varphi)\in \mathsf{PF}(m,n)} q^{\mathtt{area}(\mathrm{D},\varphi)} t^{\mathtt{dinv}(\mathrm{D},\varphi)} Q_{\mathrm{Des}(\sigma^{-1})}(X)$$

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$$\varprojlim_r C^{(r)}_{(m,n)}(x_1,\dots,x_r;q,t) = \mathcal{H}_{(m,n)}(X;q,t).$$

Thus, $C^{(r)}_{(m,n)}(x_1,\dots,x_r;q,t)$ are Schur positive, q,t-symmetric, and $x_1,\dots,x_r$-symmetric.

The Rational Shuffle Theorem

Define the **elliptic Hall algebra** $\mathcal{E}$ as the $\mathbb{C}(q,t)$-algebra generated by $\{P_{m,n} \mid m,n \in \mathbb{Z}_{\geq 0}\}$ (mod relations).

Schiffman-Vasserot gave a *geometric* action of $\mathcal{E}$ on symmetric *functions* $Sym := \mathrm{Q}(q,t)[x_1,\ldots]^{S_\infty}$.

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Studying the Khovanov-Rozansky homology of (m,n)-torus links, Gorsky-Neguț conjectured:

Rational Shuffle Theorem (Mellit)

$$P_{m,n} \cdot (1) = \mathcal{H}_{(m,n)}(X; q, t).$$

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Define the **spherical DAHA**, $\mathbb{SH}(r)$, as the spherical subalgebra of the DAHA generated by $\{P^{(r)}_{m,n} \mid m,n \in \mathbb{Z}_{\geq 0}\}$ (mod relations).

A Finite Shuffle Theorem

Theorem (Shiffmann-Vasserot)

The elliptic Hall algebra arises under the inverse limit: $\mathcal{E} \cong \varprojlim_r \mathbb{SH}(r)$ *where* $P_{(m,n)} = \varprojlim_r P^{(r)}_{(m,n)}$.

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Denote by $Sym_r := \mathbb{Q}(q,t)[x_1,\ldots,x_r]^{S_r}$ the ring of symmetric *polynomials*, recall that:

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Since $\mathcal{H}_{(m,n)}(X; q,t) = \varprojlim_r C^{(r)}_{(m,n)}(X_r; q,t)$, this yields a **finite Shuffle theorem**:

Theorem (González-Simental-Vazirani)

$$P^{(r)}_{m,n} \cdot (1) = C^{(r)}_{(m,n)}(x_1, \dots, x_r; q,t).$$

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The polynomial and geometric representations of $\mathbb{SH}(r)$ are nontrivially isomorphic (they are related via a 'truncated' plethysm).

However, describing the action directly is really hard even at $r = 1$. Maybe next time we meet I'll know the answer :) .

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THANK YOU!