Derivatives and Schubert Calculus

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Based on joint work with Zachary Hamaker (Florida), Oliver Pechenik (Waterloo), and David Speyer (Michigan).

Schubert Polynomials

Schubert Varieties

The **complete flag variety** is the quotient $\mathcal{F}\ell(n) = \mathrm{GL}(n)/B$.

There's a natural action of B on $\mathcal{F}\ell(n)$ by left multiplication. The orbits Ω_w are called **Schubert cells** and give rise to the **Bruhat decomposition**:

$$\mathcal{F}\ell(n) = \coprod_{w \in \mathcal{S}_n} \Omega_w.$$

The **Schubert varieties** are the closures of these orbits: $\mathfrak{X}_w = \overline{\Omega_w}$.

Schubert Classes

Schubert varieties give rise to **Schubert classes** σ_w in the cohomology ring $H^*(\mathcal{F}\ell(n))$. The Schubert classes form a linear basis for $H^*(\mathcal{F}\ell(n))$.

$$\sigma_u \cdot \sigma_v = \sum_{w \in \mathcal{S}_n} c_{u,v}^{w} \sigma_w$$

In general, there is no known positive, combinatorial rule for these coefficients.

The Borel Isomorphism

Thanks to Borel, there is an isomorphism

$$\Phi : H^*(\mathcal{F}\ell(n)) \to \mathbb{Z}[x_1, x_2, \ldots, x_n]/I$$

where I is the ideal generated by symmetric polynomials with no constant term.

Question: What is a "good" polynomial representative for the coset $\Phi(\sigma_w)$?

One Answer: Schubert polynomials, defined by Lascoux and Schützenberger (1982).

$$\Phi(\sigma_w) = [\mathfrak{S}_w(\mathbf{x})].$$

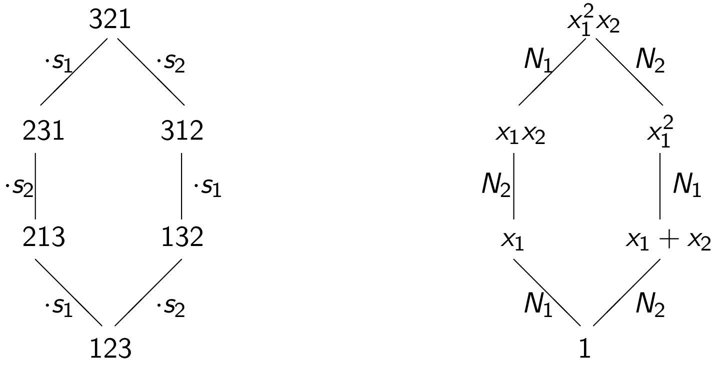
Schubert Polynomials

- Start with the **longest** permutation in $\mathcal{S}_n$

$$w_0 = n\, n-1 \ldots 1 \qquad \mathfrak{S}_{w_0} := x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$$

- The rest are defined recursively by **Newton's divided difference operators:**

$$N_i f := \frac{f - s_i \cdot f}{x_i - x_{i+1}} \qquad \text{and} \qquad \mathfrak{S}_{ws_i} := N_i \mathfrak{S}_w \text{ if } w(i) > w(i+1)$$


321
$\cdot s_1$
$\cdot s_2$
231
312
$\cdot s_2$
$\cdot s_1$
213
132
$\cdot s_1$
$\cdot s_2$
123
$x_1^2 x_2$
N_1
N_2
$x_1 x_2$
x_1^2
N_2
N_1
x_1
$x_1 + x_2$
N_1
N_2
1

The Schubert Basis

Schubert polynomials are **stable** with respect to inclusions of symmetric groups, e.g.,

$$\mathfrak{S}_{132} = \mathfrak{S}_{1324} = \mathfrak{S}_{13245} = \cdots .$$

The set $\{\mathfrak{S}_w : w \in \mathcal{S}_\infty\}$ is a basis for $\mathbb{Z}[x_1, x_2, \ldots]$.

$$\mathfrak{S}_u \mathfrak{S}_v = \sum_{w \in \mathcal{S}_\infty} c_{u,v}^w \mathfrak{S}_w$$

Derivatives of Schubert Polynomials

Question: What happens when you take “the” derivative of a Schubert polynomial?

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Example: $\mathfrak{S}_{2413} = x_1^2 x_2 + x_1 x_2^2$.

$$\frac{\partial}{\partial x_1}(\mathfrak{S}_{2413}) = 2x_1x_2 + x_2^2 \qquad \frac{\partial}{\partial x_2}(\mathfrak{S}_{2413}) = x_1^2 + 2x_1x_2$$

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$$\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)(\mathfrak{S}_{2413}) = \mathfrak{S}_{1423} + 3\mathfrak{S}_{2314}.$$

Proposition (Hamaker-Pechenik-Speyer-Weigandt, 2020)

Let $\nabla = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}$. *Then*

$$\nabla(\mathfrak{S}_w) = \sum_{s_k w < w} k \mathfrak{S}_{s_k w}.$$

Example: $\nabla(\mathfrak{S}_{2413}) = \mathfrak{S}_{1423} + 3\mathfrak{S}_{2314}$.

Proof idea: Show ∇ commutes with N_i and use this to induct with w_0 as base case.

Schur Polynomials

Schur Polynomials

- The **Schur polynomials** s_λ are indexed by **partitions** and form a basis for the **ring of symmetric polynomials**.
- Schur polynomials have a combinatorial definition as a weighted sum of **semistandard tableaux**.

Example: $\lambda = (6, 5, 3)$

$T =$

1	3	4	4	5	7
2	5	5	6	6	
3	7	8			

$$\mathtt{wt}(T) = x_1 x_2 x_3^2 x_4^2 x_5^3 x_6^2 x_7^2 x_8$$

Example: $\lambda = (3, 1)$

1	1	1
2		

1	1	2
2		

1	2	2
2		

$$s_\lambda(x_1, x_2) = x_1^2 x_2 + x_1^2 x_2^2 + x_1 x_2^3$$

Schur polynomials are Schubert polynomials

Given $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ and $k \geq \ell$, there exists some **Grassmannian** $w \in \mathcal{S}_\infty$ so that $\mathfrak{S}_w = s_\lambda(x_1, \ldots, x_k)$.

Example: $\lambda = (3, 1)$ and $k = 4$.

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4. Fill in missing numbers in increasing order $1247|356$

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In fact, all Grassmannian Schubert polynomials are Schur polynomials.

Jeu de Taquin is a sliding game on semistandard tableaux introduced by Marcel–Paul Schützenberger.

1	3	4	4	5	7
2	5	5	9	9	
6	7	8			

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6	7				

Derivatives of Schur Polynomials

- When you apply $\nabla = \sum_{i=1}^{k} \frac{\partial}{\partial x_i}$ to a symmetric polynomial, the result is symmetric.
- **Question:** What does this action look like in terms of "tableaux combinatorics"?

The Symmetric Derivative Rule

Given λ and k, the number of variables, fill each box $(i, j) \in \lambda$ with $j - i + k$.

Example: $\lambda = (4, 2, 2, 1)$ and $k = 5$

5	6	7	8
4	5		
3	4		
2			

The Symmetric Derivative Rule

Given λ and k, the number of variables, fill each box $(i, j) \in \lambda$ with $j - i + k$.

Example: $\lambda = (4, 2, 2, 1)$ and $k = 5$

5	6	7	8
4	5		
3	4		
2			

$$\nabla(s_{(4,2,2,1)}) = 8s_{(3,2,2,1)} + 4s_{(4,2,1,1)} + 2s_{(4,2,2)}$$

Corollary

Given $\nabla = \sum_{i=1}^{k} \frac{\partial}{\partial x_i}$,

$$\nabla(s_\lambda(x_1, \ldots, x_k)) = \sum_{(i,j)} (j - i + k)\, s_{\lambda-(i,j)}(x_1, \ldots, x_k)$$

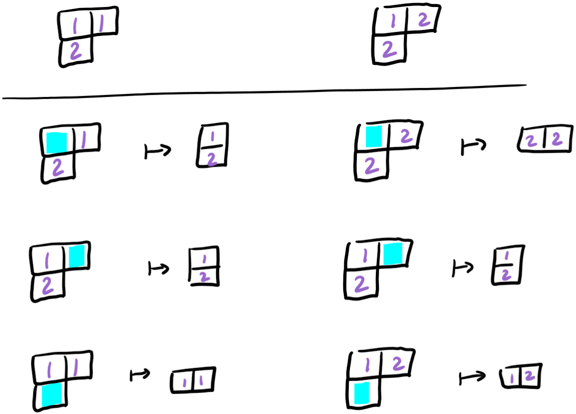
where the sum is over corners of λ.

Example: $\lambda = (4, 2, 2, 1)$ and $k = 5$

5	6	7	8
4	5		
3	4		
2			

$$\nabla(s_{(4,2,2,1)}) = 8s_{(3,2,2,1)} + 4s_{(4,2,1,1)} + 2s_{(4,2,2)}$$

Combinatorial Proof via JDT



$$\nabla s_{(2,1)}(x_1, x_2) = s_{(2)}(x_1, x_2) + 3\, s_{(1,1)}(x_1, x_2)$$

Upshot: Derivatives of Schubert (and Schur) polynomials are nice!

Goal: Use this formula to understand related combinatorial objects and constructions.

Applications of the Derivative Formula

Applications of the Schubert Derivative Formula

1. **Macdonald's Identity**
2. **Strongly Sperner Posets**
3. **Special rules for Schubert structure constants**

Macdonald's Identity

Reduced Words

Every permutation can be written as a product of simple transpositions.

Example: $w = 2413 = s_3 s_1 s_2 = s_1 s_1 s_3 s_1 s_2$.

We say an expression is **reduced** if it is as short of a factorization as possible.

The length of this factorization is the **Coxeter length** of w, denoted $\ell(w)$.

Macdonald's Identity

Theorem (Macdonald, 1991)

$$\frac{1}{\ell(w)!}\sum_{a\in R(w)} a_1 a_2 \cdots a_{\ell(w)} = \mathfrak{S}_w(1,1,\ldots,1).$$

Example

$\mathfrak{S}_{2413} = x_1^2 x_2 + x_1 x_2^2$ and $R(2413) = \{312, 132\}$.

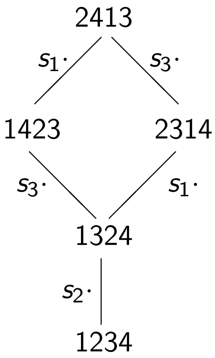
We verify

$$\frac{1}{3!}(3\cdot 1\cdot 2 + 1\cdot 3\cdot 2) = 2.$$

A (Short) Proof of Macdonald's Identity

Big idea: Chains in (left) weak Bruhat order biject with reduced words.

Example: $w = 2413$ $R(2413) = \{312, 132\}$



A (Short) Proof of Macdonald's Identity

If m is a (monic) monomial of degree k, then $\nabla^k(m) = k!$.

$\mathfrak{S}_w$ is homogeneous of degree $\ell(w)$ and therefore

$$\nabla^{\ell(w)}(\mathfrak{S}_w) = \ell(w)!\mathfrak{S}_w(1,1,\ldots,1).$$

On the other hand, since $\nabla(\mathfrak{S}_w) = \sum_{\ell(s_k w)<\ell(w)} k\mathfrak{S}_{s_k w}$,

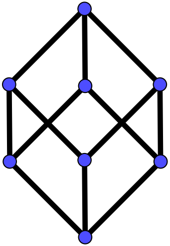
$$\nabla^{\ell(w)}(\mathfrak{S}_w) = \sum_{a\in R(w)} a_1 a_2 \cdots a_{\ell(w)}.$$

$$\ell(w)!\mathfrak{S}_w(1,1,\ldots,1) = \sum_{a\in R(w)} a_1 a_2 \cdots a_{\ell(w)}. \quad \square$$

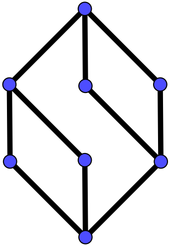
Strongly Sperner Posets

Sperner Posets

A ranked poset is **Sperner** if no antichain is larger than the largest rank level in the poset.



Sperner

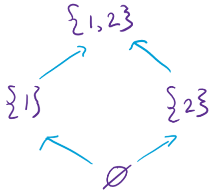


Not Sperner

The **strong Sperner property** says for all k, no union of k antichains exceeds the union of the k largest rank levels.

Theorem (Stanley 1980)

Let $P = \bigcup_{i=0}^{m} P_i$ be a ranked poset. Suppose there is an order-raising operator $U : \mathbb{Q}P \to \mathbb{Q}P$ such that if $0 \leq k < m/2$ then $U^{m-2k} : \mathbb{Q}P_k \to \mathbb{Q}P_{m-k}$ is a bijection. Then P is strongly Sperner.



$U(\emptyset) = \{1\} + \{2\}$

$U(\{1\}) = U(\{2\}) = \{1,2\}$

$U(\{1,2\}) = 0.$

$U^2(\emptyset) = 2 \cdot \{1,2\}$

The Weak Order on S_n is Strongly Sperner

- Stanley (2017) conjectured the weak order on S_n is strongly Sperner and showed that it was enough to prove the determinant of a certain matrix of **specialized Schubert polynomials** doesn't vanish.
- You can prove this via the derivative formula (Hamaker-Pechenik-Speyer-Weigandt 2020)
- Gaetz and Gao (2020) gave a separate proof first that the weak order is strongly Sperner.

Special Rules for Schubert Structure Constants

Schubert Structure Constants

Big goal: Understand the coefficients in this expansion:

$$\mathfrak{S}_u \mathfrak{S}_v = \sum_{w \in \mathcal{S}_\infty} c_{u,v}^w \mathfrak{S}_w$$

Leveraging the Product Rule

Theorem (Pechenik-Weigandt 2022)

$$\sum_{s_i u<u} i c^{w}_{s_i u,v} + \sum_{s_j v<v} j c^{w}_{u,s_j v} = \sum_{s_k w>w} k c^{s_k w}_{u,v}.$$

Proof idea: We have $\mathfrak{S}_u\mathfrak{S}_v = \sum_z c^z_{u,v}\mathfrak{S}_z$.

Apply ∇ to both sides:

$$\nabla(\mathfrak{S}_u)\mathfrak{S}_v + \mathfrak{S}_u\nabla(\mathfrak{S}_v) = \sum_z c^z_{u,v}\nabla(\mathfrak{S}_z)$$

Extract coefficients of $\mathfrak{S}_w$ from both sides.

Inverse Grassmannian Permutations

What's the easiest case? When u and v each have a unique left descent. Such permutations are called **inverse Grassmannians**.

In this case we have

$$pc^w_{s_p u,v} + qc^w_{u,s_q v} = \sum_{s_k w > w} kc^{s_k w}_{u,v}$$

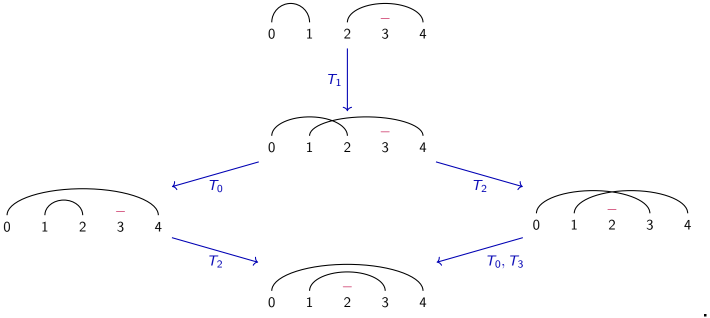
and by "stabilization tricks"

$$c^w_{s_p u,v} + c^w_{u,s_q v} = c + \sum_{s_k w > w} c^{s_k w}_{u,v}.$$

We extended a rule of Wyser in terms of (p, q)-**clans** to understand products indexed by inverse Grassmannian permutations, which lets us understand the RHS.

Backstable Clans

Let $u = 213$ and $v = 312$.



- $c^w_{u,v} = 1$ if there is a word for w from the top **clan** to the bottom.
- $c^w_{u,v}$ is zero otherwise.

Example: $c^{4123}_{u,v} = 1$

From these equations

$$pc^w_{s_p u,v} + qc^w_{u,s_q v} = \sum_{s_k w > w} kc^{s_k w}_{u,v}$$

$$c^w_{s_p u,v} + c^w_{u,s_q v} = c + \sum_{s_k w > w} c^{s_k w}_{u,v}.$$

you can extract (multiplicity free!) clan rules for $c^w_{s_p u,v}$ and $c^w_{u,s_q v}$.

Derivatives of Grothendieck Polynomials

Grothendieck Polynomials

Grothendieck polynomials are K-theoretic analogues of Schubert polynomials.

- Same initial condition:

$$\mathfrak{G}_{w_0} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}.$$

- Define $\overline{N}_i(f) = N_i\left((1 - x_{i+1})f\right)$
- If $w(i) > w(i+1)$ then $\mathfrak{G}_{ws_i} = \overline{N}_i(\mathfrak{G}_w)$.

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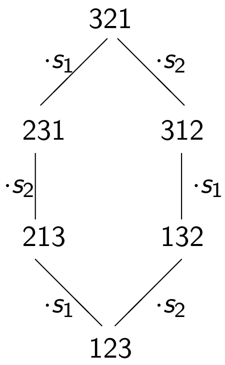
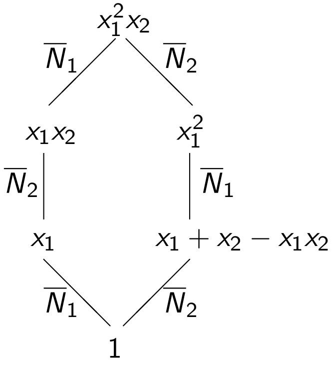
Notice: N_i might preserve degree or drop degree.

Grothendieck Polynomials for $\mathcal{S}_3$





Derivatives of Grothendieck Polynomials

Theorem (Pechenik-Speyer-Weigandt 2021)

Let $\nabla = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}$ *and* $E = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}$. *Then*

$$\left(\nabla - E + \mathrm{maj}(w^{-1})\right) \mathfrak{G}_w = \sum_{s_k w < w} k \mathfrak{G}_{s_k w},$$

where $\mathrm{maj}(w^{-1}) = \sum_{s_k w < w} k.$

Castelnuovo–Mumford Regularity

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- **Castelnuovo–Mumford regularity** is an invariant from commutative algebra that says how complicated minimal free resolutions of a graded module can be.
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- **Castelnuovo–Mumford regularity** is an invariant from commutative algebra that says how complicated minimal free resolutions of a graded module can be.
- If you happen to know the regularity, it can help computers get more info about the module faster!
- Jenna Rajchgot observed that if R/I is **Cohen-Macaulay**, the regularity of R/I is the difference between the degree of its **K-polynomial** and the **height** of I.

Schubert Determinantal Ideals

Schubert determinantal ideals I_w are generalizations of classical determinantal ideals.

Example:

$$I_{2143} = \left\langle z_{11}, \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} \right\rangle$$

Fulton (1991) showed the height of I_w is $\ell(w)$ and that R/I_w is Cohen-Macaulay.

So it's enough to find the degree of the K-polynomial!

K-polynomials via Grothendieck Polynomials

The K-polynomial of R/I_w is obtained by substituting

$$x_i \mapsto 1 - t \qquad \text{for all } i$$

into the **Grothendieck polynomial** $\mathfrak{G}_w$ (see Knutson-Miller 2004).

This change of variables is **degree preserving**.

Upshot: We just need to understand $\deg(\mathfrak{G}_w)$.

Note: Want to avoid recursive and algorithmic constructions, i.e., want simple permutation statistic.

Return to Derivatives of Grothendieck Polynomials

Theorem (Pechenik-Speyer-Weigandt 2021)

Let $\nabla = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}$ *and* $E = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}$. *Then*

$$\left(\nabla - E + \mathrm{maj}(w^{-1})\right) \mathfrak{G}_w = \sum_{s_k w < w} k \mathfrak{G}_{s_k w},$$

where $\mathrm{maj}(w^{-1}) = \sum_{s_k w < w} k$.

Upshot: Applying $\nabla - E + \mathrm{maj}(w^{-1})$ drops the degree exactly when $\deg(\mathfrak{G}_w) = \mathrm{maj}(w^{-1})$.

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Upshot: Applying $\nabla - E + \mathrm{maj}(w^{-1})$ drops the degree exactly when $\deg(\mathfrak{G}_w) = \mathrm{maj}(w^{-1})$.

Question: For which permutations does this happen?

Fireworks Permutations

A permutation is **fireworks** if the initial elements of its decreasing runs are in increasing order.

Example: 41|62|853|97

Another characterization is that these permutations are $3-12$ **pattern avoiding**.

Theorem (Pechenik-Speyer-Weigandt 2021)

$\deg(\mathfrak{G}_w) = \mathrm{maj}(w^{-1})$ *if and only if* w^{-1} *is fireworks.*

The Rajchgot Statistic

Example: $w = 71645823$

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7	1	6	4	5	8	2	3
	1	6	4	5	8	2	3
		6	4	5	8	2	3
			4	5	8	2	3
				5	8	2	3
					8	2	3
						2	3
							3

The Rajchgot Statistic

Example: $w = 71645823$

$\boxed{7}$	1	6	4	5	$\boxed{8}$	2	3
	1	6	4	5	8	2	3
		6	4	5	8	2	3
			4	5	8	2	3
				5	8	2	3
					8	2	3
						2	3
							3

The Rajchgot Statistic

Example: $w = 71645823$

$$
\begin{array}{cccccccc}
\boxed{7} & 1 & 6 & 4 & 5 & \boxed{8} & 2 & 3 \\
 & \boxed{1} & 6 & \boxed{4} & \boxed{5} & \boxed{8} & 2 & 3 \\
 & & 6 & 4 & 5 & 8 & 2 & 3 \\
 & & & 4 & 5 & 8 & 2 & 3 \\
 & & & & 5 & 8 & 2 & 3 \\
 & & & & & 8 & 2 & 3 \\
 & & & & & & 2 & 3 \\
 & & & & & & & 3
\end{array}
$$

The Rajchgot Statistic

Example: $w = 71645823$

$\boxed{7}$	1	6	4	5	$\boxed{8}$	2	3
	$\boxed{1}$	6	$\boxed{4}$	$\boxed{5}$	$\boxed{8}$	2	3
		$\boxed{6}$	4	5	$\boxed{8}$	2	3
			$\boxed{4}$	$\boxed{5}$	$\boxed{8}$	2	3
				$\boxed{5}$	$\boxed{8}$	2	3
					$\boxed{8}$	2	3
						$\boxed{2}$	$\boxed{3}$
							$\boxed{3}$

The Rajchgot Statistic

Example: $w = 71645823$

$\boxed{7}$	1	6	4	5	$\boxed{8}$	2	3
	$\boxed{1}$	6	$\boxed{4}$	$\boxed{5}$	$\boxed{8}$	2	3
		$\boxed{6}$	4	5	$\boxed{8}$	2	3
			$\boxed{4}$	$\boxed{5}$	$\boxed{8}$	2	3
				$\boxed{5}$	$\boxed{8}$	2	3
					$\boxed{8}$	2	3
						$\boxed{2}$	$\boxed{3}$
							$\boxed{3}$

The **Rajchgot statistic** on permutations is

$$\mathrm{raj}(w) = \#\text{unboxed entries} = 19.$$

Theorem (Pechenik-Speyer-Weigandt 2021)

Given $w \in \mathcal{S}_n$,

- $\deg(\mathfrak{G}_w) = \mathrm{raj}(w)$, *and*
- $\mathrm{reg}(R/I_w) = \mathrm{raj}(w) - \ell(w)$.

Example: $\mathfrak{G}_{132} = x_1 + x_2 - x_1x_2$

$$\begin{array}{ccc} \boxed{1} & \boxed{3} & 2 \\ & \boxed{3} & 2 \\ & & \boxed{2} \end{array}$$

We have $\mathrm{raj}(132) = 2 = \deg(\mathfrak{G}_{132})$.

Main Takeaway

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Differential operators are useful for extracting structure from combinatorial families of polynomials!

Thank you!