Combinatorial formulas for shifted dual stable Grothendieck polynomials

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joint work with Eric Marberg (HKUST)

FPSAC Davis

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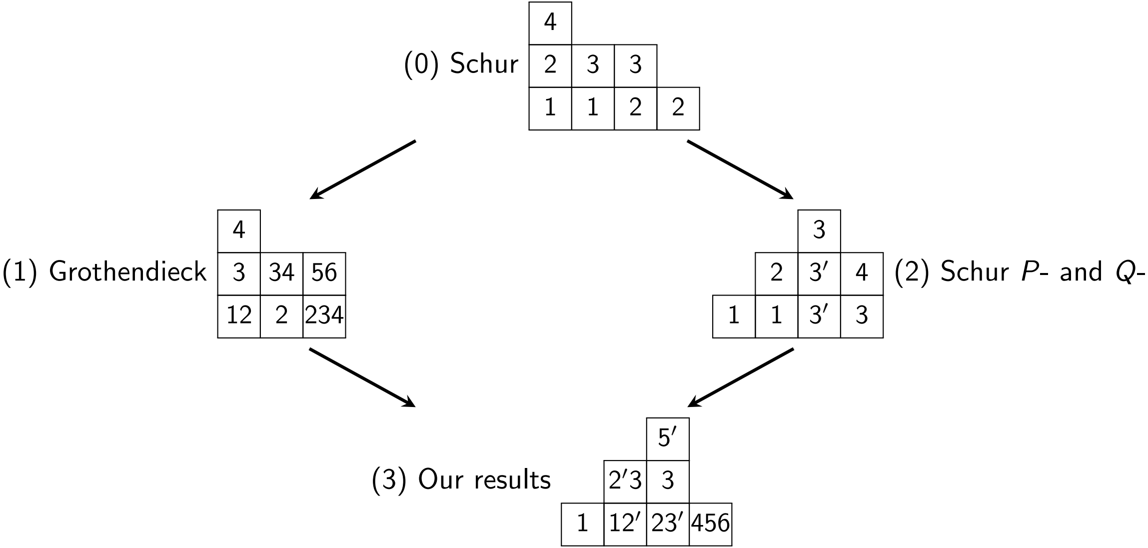
Outline of the talk

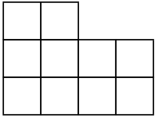
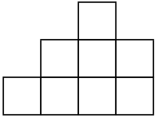


"Isn't it time for you to start studying middle-aged tableaux?" – my wife, last Tuesday

Disclaimers and conventions

In this talk, we believe:

- all symmetric functions have coefficients in $\mathbb{Z}$ (or maybe $\mathbb{Z}[\beta]$)
- all Young diagrams & tableaux (straight or shifted) are in French notation

$D_{442} =$  $SD_{431} =$ 

- it's ok to give motivation from geometry, even if the speaker doesn't really understand what "K-theory of the Lagrangian Grassmannian" means

Schur functions

Schur functions s_λ are a very special basis for the ring Sym of symmetric functions, with many different (but ultimately equivalent) definitions:

- they are the unique orthonormal $\mathbb{Z}$-basis for Sym under the Hall inner product; equiv. the unique basis that satisfies the Cauchy identity $\sum_\lambda s_\lambda(\mathbf{x})s_\lambda(\mathbf{y}) = \prod_{i,j} \frac{1}{1 - x_i y_j}$
- they are the generating functions for semistandard Young tableaux of a fixed shape

$$s_{21} = \sum_{T \in \mathrm{SSYT}(21)} \mathbf{x}^{\mathrm{wt}(T)} = \underset{x_1^2 x_2}{\begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline \end{array}} + \underset{x_1 x_2^2}{\begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array}} + \underset{2x_1x_2x_3}{\begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array}, \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array}} + \cdots$$

- they correspond to the representatives of Schubert classes in the cohomology ring of the (complex) Grassmannian
- bilaternants, representation theory, etc.

Variations on Schur functions

Take your favorite definition and tweak it:

- instead of considering cohomology, you could consider the K-theory of structure sheaves of the Grassmannian, or torus-equivariant K-theory, or other Grassmannians, etc.
- instead of considering semistandard Young tableaux, you could consider generating functions for set-valued tableaux

35	5	
12	234	5

, or "valued-set tableaux"

2	3	
1	1	2

,

or shifted, marked tableaux

		3	
	2	$3'$	4
1	1	$3'$	3

, or reverse plane partitions

2	2		
1	2	4	
1	1	1	2

,

etc., with appropriate interpretations of $\mathrm{wt}(T)$ for tableaux of each type

These often go together: e.g., the generating functions for shifted, marked tableaux (Schur Q-functions) are also representatives of cohomology classes dual to Schubert cycles in orthogonal Grassmannians and characters of irreducible representations of the queer Lie super-algebra $Q(n)$

What is lost, what is gained

One thing that is lost is self-duality: in

$$\sum_{\lambda} s_\lambda(\mathbf{x}) s_\lambda(\mathbf{y}) = \prod_{i,j} \frac{1}{1 - x_i y_j}$$

we have the same basis twice

but when we replace Schur functions with another family, we tend to get formulas like

$$\sum_{\lambda} G_\lambda(\mathbf{x}) g_\lambda(\mathbf{y}) = \prod \Big(\text{ something } \Big)$$

where G_λ are a basis for one Hopf algebra of symmetric function-like things and g_λ are a basis for a different Hopf algebra, dual to the first

(whereas Sym is self-dual)

First variation: (conjugate) (dual) (stable) Grothendieck polynomials

- Grothendieck polynomials are K-theory representatives for Schubert varieties (Lascoux–Schützenberger)
- Stable Grothendieck polynomials G_λ are certain limits of G. polys (Fomin–Kirillov)
- Stable G. polynomials are generating functions for set-valued tableaux (Buch)

$$\begin{array}{|c|c} \hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline \end{array} \qquad \begin{array}{|c|c} \hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{12} \\ \hline \end{array} \qquad \begin{array}{|c|c} \hline 3 & \\ \hline 12 & \multicolumn{1}{c|}{2} \\ \hline \end{array}, \begin{array}{|c|c} \hline 23 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array}, \begin{array}{|c|c} \hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{23} \\ \hline \end{array}$$

$$G_{21} = \sum_{T \in \mathrm{SVT}(21)} \beta^{|T|-|\lambda|} \mathbf{x}^{\mathrm{wt}(T)} = x_1^2 x_2 + \beta x_1^2 x_2^2 + 3\beta x_1 x_2^2 x_3 + \cdots$$

recover s_λ on setting $\beta = 0$

- Dual stable G. polynomials g_λ are defined by $\sum_\lambda G_\lambda(\mathbf{x}) g_\lambda(\mathbf{y}) = \prod_{i,j} \frac{1}{1 - x_i y_j}$

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- Stable G. polynomials are generating functions for set-valued tableaux (Buch) $G_\lambda = \sum_{T \in \mathrm{SVT}(\lambda)} \beta^{|T|-|\lambda|} \mathbf{x}^{\mathrm{wt}(T)}$; recover s_λ on setting $\beta = 0$
- Dual stable G. polynomials g_λ are defined by $\sum_\lambda G_\lambda(\mathbf{x}) g_\lambda(\mathbf{y}) = \prod_{i,j} \frac{1}{1 - x_i y_j}$
- The g_λ are given by a combinatorial formula

$$g_{21} = \sum_{T \in \mathrm{RPP}(21)} (-\beta)^* \mathbf{x}^{\mathrm{wt}(T)} = \underset{(-\beta)x_1^2}{\begin{array}{|c|c}\hline 1 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline\end{array}} + \underset{x_1^2 x_2}{\begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline\end{array}} + \underset{(-\beta)x_1 x_2}{\begin{array}{|c|c}\hline 1 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array}} + \underset{x_1 x_2^2}{\begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array}} + \cdots$$

summing over reverse plane partitions (with $\mathrm{wt}(T)$ recording the number of columns in which each entry appears) ...

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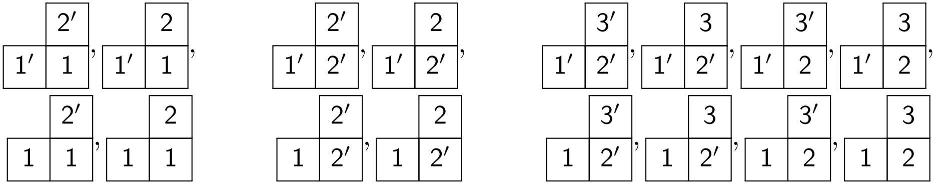
- Stable G. polynomials are generating functions for set-valued tableaux (Buch)
  $G_\lambda = \sum_{T \in \mathrm{SVT}(\lambda)} \beta^{|T|-|\lambda|} \mathbf{x}^{\mathrm{wt}(T)}$; recover s_λ on setting $\beta = 0$; $\omega(G_\lambda(\mathbf{x})) = G_{\lambda^T}(\frac{\mathbf{x}}{1-\beta\mathbf{x}})$
- Dual stable G. polynomials g_λ are defined by $\sum_\lambda G_\lambda(\mathbf{x}) g_\lambda(\mathbf{y}) = \prod_{i,j} \frac{1}{1 - x_i y_j}$
- The g_λ are given by a combinatorial formula, as are their conjugates $j_\lambda := \omega(g_{\lambda^T})$

$$j_{21} = \sum_{T \in \mathrm{BT}(21)} (-\beta)^* \mathbf{x}^{\mathrm{wt}(T)} = \begin{array}{|c|c}\hline 2 & \\ \hline \multicolumn{2}{|c|}{1} \\ \hline \end{array} \; (-\beta) x_1^2 + \begin{array}{|c|c|}\hline 2 & \multicolumn{1}{c}{} \\ \hline 1 & 1 \\ \hline \end{array} \; x_1^2 x_2 + \begin{array}{|c|c|}\hline 2 & \multicolumn{1}{c}{} \\ \hline 1 & 2 \\ \hline \end{array} \; x_1 x_2^2 + \cdots$$

summing over "valued-set tableaux" or bar tableaux (with $\mathrm{wt}(T)$ recording the number of bars in which each entry appears); recover s_λ on setting $\beta = 0$ (Lam–Pylyavskyy)

Second variation: Schur P- and Q-functions

- Schur Q-functions correspond to Schubert varieties in some orthogonal Grassmannian
- For a strict partition λ, Q_λ is the generating function for <u>shifted, marked tableaux</u>: tableaux of shifted shape λ filled with $1' < 1 < 2' < 2 < \dots$ such that both $\begin{array}{|c|}\hline i\\\hline i\\\hline\end{array}$ and $\begin{array}{|c|c|}\hline j' & j'\\\hline\end{array}$ are forbidden



$$Q_{21} = 4x_1^2x_2 + 4x_1x_2^2 + 8x_1x_2x_3 + \cdots$$

Second variation: Schur P- and Q-functions

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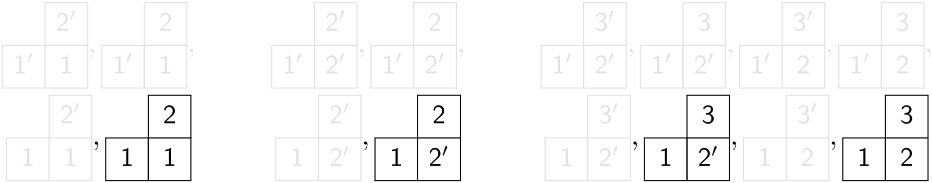
- The Schur P-functions are the same but with no $'$ on diagonal



$$P_{21} = x_1^2 x_2 + x_1 x_2^2 + 2x_1x_2x_3 + \cdots$$

so $P_\lambda = 2^{-\ell(\lambda)} Q_\lambda$

Second variation: Schur P- and Q-functions

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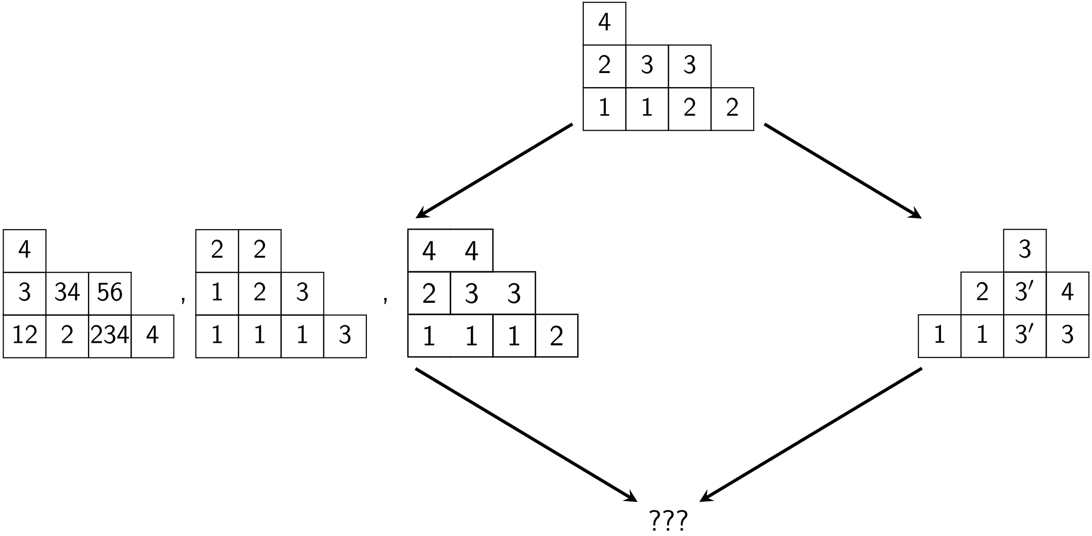
- The Schur P-functions are the same but with no $'$ on diagonal so $P_\lambda = 2^{-\ell(\lambda)} Q_\lambda$
- Shifted Cauchy identity

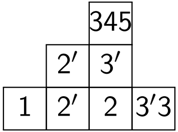
$$\sum_{\lambda \text{ strict}} Q_\lambda(\mathbf{x}) P_\lambda(\mathbf{y}) = \prod_{i,j} \frac{1 + x_i y_j}{1 - x_i y_j}$$

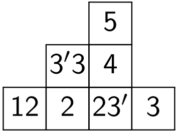
Putting everything together



Putting everything together

- Ikeda–Naruse introduced the K-theoretic Schur P- and Q-functions GP_λ and GQ_λ in their study of the K-theory ring of coherent sheaves on the Lagrangian Grassmannian (again corresponding to Schubert classes)
- They showed that these functions are given by generating functions for shifted, marked set-valued tableaux:





$$GQ_{421} = \sum_{T\in \mathrm{ShSVT}(421)} \beta^{*}\mathbf{x}^{\mathrm{wt}(T)} = \cdots + \beta^3 x_1 x_2^3 x_3^4 x_4 x_5 + \beta^3 x_1 x_2^3 x_3^4 x_4 x_5 + \cdots$$

(and GP_λ the same with no $'$ on the diagonal)

- Note that $GP_\lambda \neq 2^{-\ell(\lambda)} GQ_\lambda$!!
- Recover Q_λ and P_λ on setting $\beta = 0$

Putting everything together

- Ikeda–Naruse introduced the K-theoretic Schur P- and Q-functions GP_λ and GQ_λ in their study of the K-theory ring of coherent sheaves on the Lagrangian Grassmannian (again corresponding to Schubert classes)
- They showed that these functions are given by generating functions for shifted, marked set-valued tableaux
- Recover Q_λ and P_λ on setting $\beta = 0$
- Nakagawa–Naruse defined dual K-theoretic Schur P- and Q-functions gp_λ and gq_λ by the following Cauchy identity:

$$\sum_\lambda GQ_\lambda(\mathbf{x})gp_\lambda(\mathbf{y}) = \sum_\lambda GP_\lambda(\mathbf{x})gq_\lambda(\mathbf{y}) = \prod_{i,j\geq 1} \frac{1-\overline{x_i}y_j}{1-x_iy_j} \quad \text{where } \overline{x} := \frac{-x}{1+\beta x}.$$

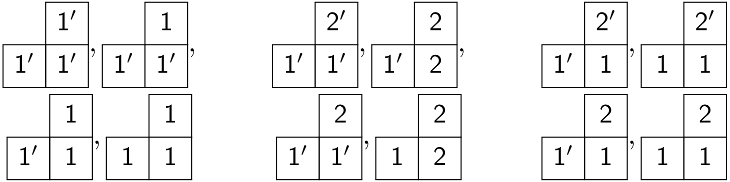
  They conjectured formulas for gp_λ and gq_λ as generating functions for shifted, marked reverse plane partitions, and Chiu–Marberg conjectured formulas for $\omega(gp_\lambda)$ and $\omega(gq_\lambda)$ as generating functions for shifted, marked bar tableaux

Main theorem

GQ_λ, GP_λ are generating functions for shifted, marked set-valued tableaux
gp_λ and gq_λ defined by Cauchy identity $\sum_\lambda GQ_\lambda(\mathbf{x})gp_\lambda(\mathbf{y}) = \sum_\lambda GP_\lambda(\mathbf{x})gq_\lambda(\mathbf{y}) = \prod(\cdots)$

Theorem (L–Marberg)

gq_λ and gp_λ are generating functions for shifted, marked reverse plane partitions



$$gq_{21} = \quad 4(-\beta)x_1^2 \quad + \quad 4(-\beta)x_1x_2 \quad + \quad 4x_1^2x_2 \quad + \quad \cdots$$

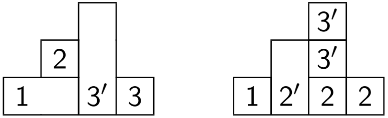
(gp requires all diagonal entries primed)

Main theorem

GQ_λ, GP_λ are generating functions for shifted, marked set-valued tableaux
gp_λ and gq_λ defined by Cauchy identity $\sum_\lambda GQ_\lambda(\mathbf{x})gp_\lambda(\mathbf{y}) = \sum_\lambda GP_\lambda(\mathbf{x})gq_\lambda(\mathbf{y}) = \prod(\cdots)$

Theorem (L–Marberg)

gq_λ and gp_λ are generating functions for shifted, marked reverse plane partitions $gq_\lambda = \sum_{T \in \mathrm{ShRPP}(21)} (-\beta)^* \mathbf{x}^{\mathrm{wt}(T)}$, and their conjugates $jp_\lambda = \omega(gp_\lambda)$ and $jq_\lambda = \omega(gq_\lambda)$ are generating functions for shifted, marked bar tableaux



$$jq_{421} = \sum_{T \in \mathrm{ShBT}(421)} (-\beta)^* \mathbf{x}^{\mathrm{wt}(T)} = \cdots + \quad (-\beta)^3 x_1 x_2 x_3^2 \quad + \quad (-\beta) x_1 x_2^3 x_3^2 \quad + \quad \cdots$$

(jp requires all diagonal entries unprimed)

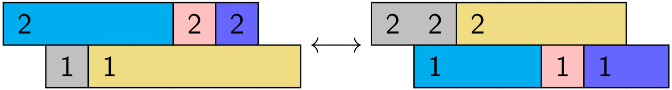
Proof ideas

GQ_λ, GP_λ are generating functions for shifted, marked set-valued tableaux
gp_λ and gq_λ defined by Cauchy identity $\sum_\lambda GQ_\lambda(\mathbf{x})gp_\lambda(\mathbf{y}) = \sum_\lambda GP_\lambda(\mathbf{x})gq_\lambda(\mathbf{y}) = \prod(\cdots)$

Theorem (L–Marberg)

gq_λ and gp_λ are generating functions for shifted, marked reverse plane partitions, and their conjugates $jp_\lambda = \omega(gp_\lambda)$ and $jq_\lambda = \omega(gq_\lambda)$ are generating functions for shifted, marked bar tableaux.

- Generalize to skew (shifted) shapes $SD_\lambda \setminus SD_\mu$, polynomials $gq_{\lambda/\mu}$, $gp_{\lambda/\mu}$, $jq_{\lambda/\mu}$, $jp_{\lambda/\mu}$
- Totally unclear that the combinatorial formulas define symmetric functions; we prove this by an appropriate version of Bender–Knuth involutions, one piece of which looks like this:



- Do everything explicitly when $\lambda = (r)$ is a one-part partition
- Establish Pieri rules by a combination of combinatorial and algebraic reasoning (using the Cauchy identity), and then declare victory by induction

A consequence

- As generating functions for shifted, marked set-valued tableaux

		345	
	2′	3′	
1	2′	2	3′3

		5	
	3′3	4	
12	2	23′	3

$$GQ_{421} = \sum_{T \in \mathrm{ShSVT}(421)} \beta^{*} \mathbf{x}^{\mathrm{wt}(T)} = \cdots + \beta^3 x_1 x_2^3 x_3^4 x_4 x_5 + \beta^3 x_1 x_2^3 x_3^4 x_4 x_5 + \cdots$$

GQ_λ, GP_λ have terms of arbitrarily large **x**-degree. Consequently it is not clear that $GQ_\lambda \cdot GQ_\mu$ is a finite linear combination of GQ_νs (and ditto for GP). In other words: not clear that the linear span is a ring. Conjecture (Ikeda–Naruse): they are rings

A consequence

- As generating functions for shifted, marked set-valued tableaux

$$GQ_{421} = \sum_{T\in \mathrm{ShSVT}(421)} \beta^{*}\mathbf{x}^{\mathrm{wt}(T)} = \cdots + \underset{\beta^{3}x_1x_2^3x_3^4x_4x_5}{\begin{array}{cccc} & & 345 & \\ & 2' & 3' & \\ 1 & 2' & 2 & 3'3 \end{array}} + \underset{\beta^{3}x_1x_2^3x_3^4x_4x_5}{\begin{array}{cccc} & & 5 & \\ & 3'3 & 4 & \\ 12 & 2 & 23' & 3 \end{array}} + \cdots$$

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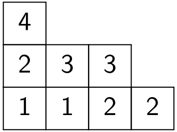
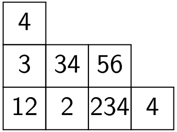
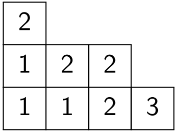
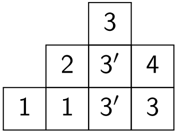
- Case of GP (but *not* GQ) done by Clifford–Thomas–Yong, including explicit Littlewood–Richardson rule for multiplying $GP_\lambda \cdot GP_\mu$
- Combined with work of Chiu–Marberg, our theorem implies GQ_λ generate a ring
- However, it does *not* give a Littlewood–Richardson rule for multiplying GQs. Eric's paper *Shifted combinatorial Hopf algebras from K-theory* `arXiv:2211.01092` gives a comprehensive account of all these objects, and open questions









Thanks for listening!