Chromatic quasisymmetric functions and noncommutative $P$-symmetric functions

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1 History of chromatic quasisymmetric functions

2 An equivalence relation on words

3 Noncommutative $P$-symmetric functions
History of chromatic quasisymmetric functions

An equivalence relation on words

Noncommutative $P$-symmetric functions
Natural unit interval orders

Fix a positive integer $n$. Let $m = (m_1, \ldots, m_n)$ be a weakly increasing sequence such that $i \leq m_i \leq n$ for each $i$. Then the natural unit interval order $P(m)$ corresponding to $m$ is a poset on $[n] = \{1, 2, \ldots, n\}$ with the ordering $<_P$ given by

$$i <_P j \text{ if and only if } m_i < j.$$
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$$i <_P j \text{ if and only if } m_i < j.$$ 

Example

Let $P = P(2, 4, 5, 5, 5)$. 

```
    3
   / \  \
 4   5
  |   |
 /   /  \
1   2
```
Words on $[n]$

Let $P$ be a poset on $[n]$, and $w = w_1 w_2 \cdots w_d$ a word on $[n]$.

- $w$ is of type $\mu = (\mu_1, \ldots, \mu_n)$ if $i$ appears $\mu_i$ times in $w$. Denote by $W(\mu)$ the set of words of type $\mu$.
- $i$ is a $P$-descent of $w$ if $w_i \succ_P w_{i+1}$, and

$$\text{Des}_P(w) = \{ i \in [d-1] \mid w_i \succ_P w_{i+1} \}.$$  

- $(i < j)$ is a $P$-inversion pair of $w$ if $w_i$ and $w_j$ are incomparable in $P$, and $w_i \succ w_j$ in the natural order on $P$, and $\text{inv}_P(w)$ denotes the number of $P$-inversion pairs.
Chromatic symmetric functions

In 1995, Stanley introduced *chromatic symmetric functions*, which are a generalization of chromatic polynomials. For a poset $P$,

$$\omega X_P(x) = \sum_{w \in W(1^n)} F_{n,\text{Des}_P(w)}(x).$$

**The Stanley–Stembridge conjecture**

*For a natural unit interval order $P$, let*

$$\omega X_P(x) = \sum_{\lambda} c_{\lambda} h_{\lambda}(x),$$

*then $c_{\lambda} \geq 0$ for all $\lambda$.***
In 2016, Shareshian and Wachs introduced a quasisymmetric generalization of a chromatic symmetric function: for a poset $P$,

$$\omega X_P(x, q) = \sum_{w \in W(1^n)} q^{\text{inv}_P(w)} F_{n, \text{Des}_P(w)}(x).$$

They proved that $\omega X_P(x, q)$ is also symmetric when $P$ is a natural unit interval order.
Theorem (Gasharov 96’, Shareshian–Wachs 16’)

For a natural unit interval order $P$, $\omega X_P(x, q)$ is Schur positive.
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For a natural unit interval order $P$, $\omega X_P(x, q)$ is Schur positive.

The refined Stanley–Stembridge conjecture (Shareshian–Wachs 16’)

For a natural unit interval order $P$, let

$$\omega X_P(x, q) = \sum_{\lambda} c_\lambda(q) h_\lambda(x),$$

then $c_\lambda(q) \in \mathbb{N}[q]$ for all $\lambda$. 
Advantage of the quasisymmetric refinement

Obviously, the refined conjecture is stronger than the original conjecture. On the other hand, the $P$-inversion statistic gives us a hint for the $e$-positivity conjecture. Indeed, we can cluster words along with their $P$ inversions, and thus concentrate certain words rather than the whole words.
Advantage of the quasisymmetric refinement

Obviously, the refined conjecture is stronger than the original conjecture. On the other hand, the $P$-inversion statistic gives us a hint for the $e$-positivity conjecture. Indeed, we can cluster words along with their $P$ inversions, and thus concentrate certain words rather than the whole words.

In addition, the quasisymmetric refinement gives us a bridge between the chromatic symmetric functions and the cohomology of Hessenberg varieties, and the $q$-grading on words plays the role of the cohomology degree.
Advantage of the quasisymmetric refinement

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In addition, the quasisymmetric refinement gives us a bridge between the chromatic symmetric functions and the cohomology of Hessenberg varieties, and the $q$-grading on words plays the role of the cohomology degree.

Question

Can we refine the chromatic quasisymmetric functions $\omega X_P(x, q)$?
History of chromatic quasisymmetric functions

An equivalence relation on words

Noncommutative $P$-symmetric functions
Fix a natural unit interval order $P$ on $[n]$. For $\mu \in \mathbb{N}^n$, define the 
(multi-)chromatic quasisymmetric function $\omega X_P(x, q; \mu)$ to be

$$\omega X_P(x, q; \mu) = \sum_{w \in W(\mu)} q^{\text{inv}_P(w)} F_{d, \text{Des}_P(w)}(x),$$

where $d = \mu_1 + \cdots + \mu_n$.

By definition, $\omega X_P(x) = \omega X_P(x, 1; (1^n))$ and $\omega X_P(x, q) = \omega X_P(x, q; (1^n))$. 

**Chromatic quasisymmetric functions**
An equivalence relation on words

For $\mu \in \mathbb{N}^n$, let $\Gamma_\mu$ be a graph whose vertex set is $W(\mu)$ and two words $w$ and $v$ are adjacent if and only if $(w, v)$ are of one of the following forms:

\[
\begin{align*}
\text{(i) } & \quad w = \cdots ac\cdots \quad \text{if } a <_P c. \\
& \quad v = \cdots ca\cdots
\end{align*}
\]

\[
\begin{align*}
\text{(ii) } & \quad w = \cdots acb\cdots \quad \text{if } a < b < c, \ a \not<_P b \not<_P c, \text{ and } a <_P c. \\
& \quad v = \cdots bac\cdots
\end{align*}
\]

For $w, v \in W(\mu)$, we say $w \sim v$ if they belong to the same connected component in $\Gamma_\mu$. 
Example

Let $P = P(2, 3, 3)$ and $\mu = (1, 1, 2)$. Then $W(\mu)$ consists of 12 words.

\begin{align*}
1233 & \quad 1323 & \quad 3123 & \quad 1332 & \quad 3132 & \quad 3312 \\
2133 & \quad 2313 & \quad 2331 & \quad 3213 & \quad 3231 & \quad 3321
\end{align*}
Example

Let $P = P(2, 3, 3)$ and $\mu = (1, 1, 2)$. Then $W(\mu)$ consists of 12 words.

1233  1323  3123  1332  3132  3312
2133  2313  2331  3213  3231  3321
Example

Let $P = P(2, 3, 3)$ and $\mu = (1, 1, 2)$. Then $W(\mu)$ consists of 12 words.

\[
\begin{array}{cccc}
1233 & 1323 & 3123 & 1332 & 3132 & 3312 \\
2133 & 2313 & 2331 & 3213 & 3231 & 3321 \\
\end{array}
\]
Proposition (H.)

For \( w, v \in W(\mu) \),

\[ w \sim v \implies \text{inv}_P(w) = \text{inv}_P(v), \]

but the converse does not hold.

Let \( \Gamma \) be a connected component in \( \Gamma_{\mu} \), and

\[ K_{\Gamma}(x) = \sum_{w \in \Gamma} F_{d, \text{Des}_P(w)}(x). \]

Theorem (H.)

For a connected component \( \Gamma \), \( K_{\Gamma}(x) \) is symmetric. In particular, \( \omega X_P(x, q; \mu) \) is symmetric.
A refinement of the refined Stanley–Stembridge conjecture

Conjecture 1 (Stanley–Stembridge 93’, Stanley 95’)

\[ \omega X_P(x, 1; (1^n)) \text{ is } h\text{-positive}. \]
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\[ \omega X_P(x, 1; (1^n)) \text{ is } h\text{-positive}. \]

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\[ \omega X_P(x, q; (1^n)) \text{ is } h\text{-positive}. \]
Conjecture 1 (Stanley–Stembridge 93’, Stanley 95’)

\( \omega X_P(x, 1; (1^n)) \) is \( h \)-positive.

Conjecture 2 (Shareshian–Wachs 16’)

\( \omega X_P(x, q; (1^n)) \) is \( h \)-positive.

Conjecture 3 (H.)

For each connected component \( \Gamma \), \( K_\Gamma(x) \) is \( h \)-positive. In particular, 
\( \omega X_P(x, q; \mu) \) is \( h \)-positive.
Example

Let \( P = P(3, 3, 4, 4) \) and \( \mu = (1, 1, 1, 1) \).

\[
\begin{array}{cccccc}
2134 & 4213 & 2413 & 3412 & 4312 \\
4123 & 2143 & 3142 & 3124 & 3421 \\
1234 & | & | & | & |
1423 & 1342 & 2314 & 1432 & 2431 & 3241 \\
1243 & | & | & | & |
1324 & 2341 & 4132 & 4231 & 3214 & 4321
\end{array}
\]
Example

Let $P = P(3, 3, 4, 4)$ and $\mu = (1, 1, 1, 1)$.

\begin{verbatim}
2134  4213  2413  3412  4312
|     |     |     |
4123  2143  3142  3124  3421
|     |     |     |
1234  |     |     |     |
1423  1342  2314  1432  2431  3241
|     |     |     |     |     |
1243  1324  2341  4132  4231  3214
\end{verbatim}

Conjecture 1
Example

Let $P = P(3, 3, 4, 4)$ and $\mu = (1, 1, 1, 1)$.
Example
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Conjecture 3
An equivalence relation on words

Example

Let $P = P(3, 3, 4, 4)$ and $\mu = (1, 1, 1, 1)$.

Conjecture 3 $\implies$ Conjecture 2 $\implies$ Conjecture 1
Natural unit interval orders vs. \((3 + 1)\)-free posets

Natural unit interval orders are characterized by \((3 + 1)\)- and \((2 + 2)\)-freeness.

\[
\{\text{natural unit interval orders}\} \subsetneq \{\text{(3 + 1)-free posets}\}
\]

The original Stanley–Stembridge conjecture is about \((3 + 1)\)-free posets, not natural unit interval orders.

The graph structure on \(W(\mu)\) and the equivalence relation can be defined for \((3 + 1)\)-free posets. Hence Conjecture 3 can be extended to \((3 + 1)\)-free posets. See Blasiak–Eriksson–Pylyavskyy–Siegl 22’.
1 History of chromatic quasisymmetric functions
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3 Noncommutative $P$-symmetric functions
Let $\mathcal{U} = \mathbb{Z}\langle u_1, \ldots, u_n \rangle$ be the free associative $\mathbb{Z}$-algebra generated by $\{u_1, \ldots, u_n\}$. For simplicity we often write $u_w = u_{w_1} u_{w_2} \cdots u_{w_d}$ for a word $w = w_1 w_2 \cdots w_d$ on $[n]$.

Let $\mathcal{I}_P$ be the 2-sided ideal of $\mathcal{U}$ generated by the following elements:

\begin{align*}
    u_a u_c - u_c u_a &\quad (a <_P c), \\
    u_a u_c u_b - u_b u_a u_c &\quad (a < b < c, \; a \not<_P b \not<_P c \; \text{and} \; a <_P c).
\end{align*}
Definition

For $k \geq 1$, the noncommutative $P$-elementary symmetric function $e_k(u)$ is

$$e_k(u) = \sum_{i_1 > P i_2 > P \cdots > P i_k} u_{i_1} u_{i_2} \cdots u_{i_k} \in \mathcal{U}.$$ 

Example

Let $P = P(2, 4, 5, 5, 5)$. Then $e_2(u) = u_3 u_1 + u_4 u_1 + u_5 u_1 + u_5 u_2$, and $e_3(u) = 0$. 
Definition

For $k \geq 1$, the noncommutative $P$-elementary symmetric function $\varepsilon_k(u)$ is

$$\varepsilon_k(u) = \sum_{i_1 > P i_2 > P \cdots > P i_k} u_{i_1} u_{i_2} \cdots u_{i_k} \in \mathcal{U}.$$ 

Example

Let $P = P(2, 4, 5, 5, 5)$. Then $\varepsilon_2(u) = u_3 u_1 + u_4 u_1 + u_5 u_1 + u_5 u_2$, and $\varepsilon_3(u) = 0$.

Definition

For $k \geq 1$, define the noncommutative $P$-complete homogeneous symmetric function $\eta_k(u)$ by

$$\eta_k(u) = \sum_{i_1 \not> P i_2 \not> P \cdots \not> P i_k} u_{i_1} u_{i_2} \cdots u_{i_k} \in \mathcal{U}.$$
Theorem (H.)

For $k, \ell \geq 1$, $e_k(u)$ and $e_\ell(u)$ commute with each other modulo $\mathcal{I}_P$, that is,

$$e_k(u)e_\ell(u) \equiv e_\ell(u)e_k(u) \pmod{\mathcal{I}_P}.$$
The noncommutative $P$-Cauchy product

Recall the Cauchy product:

$$C(x, y) = \prod_{i,j} \frac{1}{1 - x_i y_j} = \prod_{j \geq 1} \sum_{\ell \geq 0} x_j^\ell h_\ell(y).$$
The noncommutative $P$-Cauchy product

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Definition (H.)

The noncommutative $P$-Cauchy product $\Omega(x, u)$ is

$$\Omega(x, u) = \prod_{j \geq 1} \sum_{\ell \geq 0} x_j^\ell h_\ell(u) \in \mathcal{U}[[x]].$$

Observing the definition of $\Omega(x, u)$ carefully, we have

$$\Omega(x, u) = \sum_w F_{d, \text{Des}_P}(w)(x) u_w,$$

where $w$ ranges over all words on $[n]$ and $d$ is the length of $w$. 
Let
\[ U^* = \mathbb{Z}\langle w \mid w \text{ is a word on } [n]\rangle \quad \text{and} \quad U_q^* := \mathbb{Z}[q] \otimes U^*, \]
and define a bilinear pairing between \( U \) and \( U^* \) (and \( U_q^* \)) by
\[ \langle u_w, v \rangle := \delta_{w,v} \text{ for words } w \text{ and } v. \]

Let
\[ \gamma_\Gamma = \sum_{w \in \Gamma} w \in U^* \quad \text{and} \quad \gamma_\mu = \sum_{w \in W(\mu)} q^{\text{inv}_P(w)} w \in U_q^*. \]

Then by definition,
\[ K_\Gamma(x) = \langle \Omega(x, u), \gamma_\Gamma \rangle \quad \text{and} \quad \omega X_P(x, q; \mu) = \langle \Omega(x, u), \gamma_\mu \rangle. \]
Theorem (H.)

Suppose that

\[ \Omega(x, u) \equiv \sum_{\lambda} g_\lambda(x)f_\lambda(u) \mod I_p[[x]]. \]

Let \( K_\Gamma(x) = \sum_{\lambda} r_{\lambda, \Gamma} g_\lambda(x) \) and \( \omega X_P(x, q; \mu) = \sum_{\lambda} r_\lambda(q)g_\lambda(x) \). Then for any \( \lambda \),

\[ r_{\lambda, \Gamma} = \langle f_\lambda(u), \gamma_{\Gamma} \rangle \quad \text{and} \quad r_\lambda(q) = \langle f_\lambda(u), \gamma_{\mu} \rangle. \]

In particular, if \( f_\lambda(u) \) has a positive monomial expression modulo \( I_P \) for all \( \lambda \), then \( K_\Gamma(x) \) and \( \omega X_P(x, q; \mu) \) are \( g \)-positive.
Recall that for a partition $\lambda$,

$$s_\lambda(x) = \det(e_{\lambda'_i - i + j}(x))_{i,j} = \sum_{\sigma \in S_m} \text{sgn}(\sigma)e_{\lambda'_1 - 1 + \sigma_1}(x) \cdots e_{\lambda'_m - m + \sigma_m}(x),$$

where $\lambda'$ is the conjugate of $\lambda$ and $m = \lambda_1$. 

Definition (H.)

For a partition $\lambda$, the noncommutative $P$-Schur function $J_\lambda(u)$ is

$$J_\lambda(u) := \sum_{\sigma \in S_m} \text{sgn}(\sigma)e_{\lambda'_1 - 1 + \sigma_1}(u) \cdots e_{\lambda'_m - m + \sigma_m}(u).$$

Proposition (H.)

We have $\Omega(x, u) \equiv \sum \lambda s_\lambda(x) J_\lambda(u) \mod I_P[[x]]$. 
Noncommutative $P$-Schur functions

Recall that for a partition $\lambda$,

$$s_\lambda(x) = \det(e_{\lambda'_i-i+j}(x))_{i,j} = \sum_{\sigma \in \mathcal{S}_m} \text{sgn}(\sigma)e_{\lambda'_1-1+\sigma_1}(x) \cdots e_{\lambda'_m-m+\sigma_m}(x),$$

where $\lambda'$ is the conjugate of $\lambda$ and $m = \lambda_1$.

Definition (H.)

For a partition $\lambda$, the noncommutative $P$-Schur function $\mathcal{J}_\lambda(u)$ is

$$\mathcal{J}_\lambda(u) := \sum_{\sigma \in \mathcal{S}_m} \text{sgn}(\sigma)e_{\lambda'_1-1+\sigma_1}(u) \cdots e_{\lambda'_m-m+\sigma_m}(u).$$
Noncommutative $P$-Schur functions

Recall that for a partition $\lambda$,

$$
s_\lambda(x) = \det(e_{\lambda'_i - i + j}(x))_{i,j} = \sum_{\sigma \in S_m} \text{sgn}(\sigma) e_{\lambda'_1 - 1 + \sigma_1}(x) \cdots e_{\lambda'_m - m + \sigma_m}(x),
$$

where $\lambda'$ is the conjugate of $\lambda$ and $m = \lambda_1$.

Definition (H.)

For a partition $\lambda$, the noncommutative $P$-Schur function $\widehat{J}_\lambda(u)$ is

$$
\widehat{J}_\lambda(u) := \sum_{\sigma \in S_m} \text{sgn}(\sigma) e_{\lambda'_1 - 1 + \sigma_1}(u) \cdots e_{\lambda'_m - m + \sigma_m}(u).
$$

Proposition (H.)

We have

$$
\Omega(x, u) \equiv \sum_{\lambda} s_\lambda(x) \widehat{J}_\lambda(u) \mod \mathcal{I}_P[[x]].
$$
Definition

For a partition $\lambda$, a \textit{semistandard $P$-tableau} of shape $\lambda$ is a filling of Young diagram of shape $\lambda$ with $[n]$ satisfying that

(i) each row is non-$P$-decreasing from left to right, and

(ii) each column is $P$-increasing from top to bottom.

• $T$ is of \textit{type} $\mu = (\mu_1, \ldots, \mu_n)$ if each $i \in [n]$ appears $\mu_i$ times in $T$.
• $w(T) =$ the column reading word of $T$.
• $T_P(\lambda) =$ the set of all semistandard $P$-tableaux of shape $\lambda$. 

Noncommutative $P$-symmetric functions

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Theorem (H.)

For a partition $\lambda$, we have

$$\tilde{J}_\lambda(u) \equiv \sum_{T \in \mathcal{T}_P(\lambda)} u_{w(T)} \mod \mathcal{I}_P.$$ 

Consequently,

$$\omega X_P(x, q; \mu) = \sum_T q^{\text{inv}_P(w(T))} s_{\text{sh}(T)}(x),$$

where $T$ ranges over all semistandard $P$-tableaux of type $\mu$ and $\text{sh}(T)$ denotes the shape of $T$. 

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Example

Let $P = P(2, 3, 3)$ and $\mu = (1, 1, 2)$. Then

\[
\begin{align*}
1233 & \quad 1323 & \quad 3123 & \quad 1332 & \quad 3132 & \quad 3312 \\
2133 & \quad 2313 & \quad 2331 & \quad 3213 & \quad 3231 & \quad 3321
\end{align*}
\]
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1233 & \quad 1323 \quad 3123 \quad 1332 \quad 3132 \quad 3312 \\
2133 & \quad 2313 \quad 2331 \quad 3213 \quad 3231 \quad 3321
\end{align*}

\begin{align*}
s_4(x) & \quad 2s_4(x) \quad 2s_4(x) \quad s_4(x) \\
\begin{bmatrix} 1 & 2 & 3 & 3 \\ 2 & 1 & 3 & 3 \end{bmatrix} & \begin{bmatrix} 1 & 3 & 2 & 3 \\ 3 & 2 & 1 & 3 \end{bmatrix} & \begin{bmatrix} 1 & 3 & 3 & 2 \\ 3 & 3 & 2 & 1 \end{bmatrix}
\end{align*}
Example

Let $P = P(2, 3, 3)$ and $\mu = (1, 1, 2)$. Then

$$
\begin{array}{cccccccc}
1233 & 1323 & 3123 & 1332 & 3132 & 3312 \\
2133 & 2313 & 2331 & 3213 & 3231 & 3321 \\
\end{array}
$$

$$
\begin{array}{cccc}
s_4(x) & 2s_4(x) & 2s_4(x) & s_4(x) \\
1 & 2 & 3 & 1 \\
3 & 3 & 1 & 3 \\
\end{array}
$$
Example

Let $P = P(2, 3, 3)$ and $\mu = (1, 1, 2)$. Then

$$
\begin{array}{cccc}
1233 & 1323 & 3123 & 1332 \\
| & | & | & |
2133 & 2313 & 2331 & 3213 \\
3213 & 3231 & 3321 & 3312
\end{array}
$$

$$
\begin{array}{cccc}
s_4(x) & 2s_4(x) + s_{3,1}(x) & 2s_4(x) + s_{3,1}(x) & s_4(x)
\end{array}
$$

$$
\begin{array}{ccc}
1 & 2 & 3 \\
3 & & \\
3 & &
\end{array}
$$

$$
\begin{array}{ccc}
1 & 3 & 2 \\
3 & & \\
3 & &
\end{array}
$$
Example

Let $P = P(2, 3, 3)$ and $\mu = (1, 1, 2)$. Then

\[
\begin{array}{cccc}
1233 & 1323 & 3123 & 1332 \\
2133 & 2313 & 2331 & 3213 \\
2313 & 2331 & 3213 & 3321 \\
\end{array}
\]

\[
s_4(x) & 2s_4(x) + s_{3,1}(x) & 2s_4(x) + s_{3,1}(x) & s_4(x)
\]

\[
\begin{array}{ccc}
1 & 2 & 3 \\
3 & & \\
\end{array}
\quad
\begin{array}{ccc}
1 & 3 & 2 \\
3 & & \\
\end{array}
\]

\[
\omega X_P(x, q; \mu) = (q^3 + 2q^2 + 2q + 1)s_4(x) + (q^2 + q)s_{3,1}(x).
\]
Noncommutative $P$-monomial symmetric functions

Let $(N_{\lambda,\mu})$ be the transition matrix between monomial symmetric functions $m_\lambda(x)$ and elementary symmetric functions $e_\mu(x)$, i.e.,

$$m_\lambda(x) = \sum_{\mu} N_{\lambda,\mu} e_\mu(x).$$

**Definition (H.)**

For a partition $\lambda$, the **noncommutative $P$-monomial symmetric function** $m_\lambda(u)$ is

$$m_\lambda(u) := \sum_{\mu} N_{\lambda,\mu} e_\mu(u).$$
Proposition (H.)

We have

\[ \Omega(x, u) \equiv \sum_{\lambda} h_\lambda(x)m_\lambda(u) \mod \mathcal{I}_P[[x]]. \]

Therefore, finding a positive monomial expression of \( m_\lambda(u) \) is equivalent to finding a positive combinatorial interpretation of the coefficients of the \( h \)-expansion of \( K_\Gamma(x) \), which proves Conjecture 3.
Proposition (H.)

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\[ \Omega(x, u) \equiv \sum_{\lambda} h_\lambda(x) m_\lambda(u) \mod \mathcal{I}_P[[x]]. \]

Therefore, finding a positive monomial expression of \( m_\lambda(u) \) is equivalent to finding a positive combinatorial interpretation of the coefficients of the \( h \)-expansion of \( K_\Gamma(x) \), which proves Conjecture 3.

Theorem (H.)

For a partition \( \lambda \) of hook shape or 2-column shape, \( m_\lambda(u) \) has a positive monomial expression modulo \( \mathcal{I}_P \).

In particular, let \( K_\Gamma(x) = \sum_\lambda c_{\lambda,\Gamma} h_\lambda(x) \), then \( c_{\lambda,\Gamma} \geq 0 \) if \( \lambda \) is of hook shape or 2-column shape.
Further directions

(1) Equivalence relation on the Hessenberg varieties. The $q$-grading (the $P$-inversion statistic) on the chromatic quasisymmetric functions has a geometric meaning: the cohomology degree of the Hessenberg variety. The equivalence relation refines the statistic, and the equivalence classes define Schur positive symmetric functions. Is there a geometric meaning of the equivalence classes?
Further directions

(1) Equivalence relation on the Hessenberg varieties. The $q$-grading (the $P$-inversion statistic) on the chromatic quasisymmetric functions has a geometric meaning: the cohomology degree of the Hessenberg variety. The equivalence relation refines the statistic, and the equivalence classes define Schur positive symmetric functions. Is there a geometric meaning of the equivalence classes?

(2) Equivalence relation on unicellular LLT polynomials. There is a plethystic formula between chromatic quasisymmetric functions and unicellular LLT polynomials:

$$\text{LLT}_P(x, q) = (q - 1)^n X_P[x/(q - 1); q].$$

Is there a similar equivalence relation on monomials of LLT polynomials?
References I


Thank you!