Chromatic quasisymmetric functions and noncommutative P-symmetric functions

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Formal Power Series and Algebraic Combinatorics 2023
July 17, 2023

1. History of chromatic quasisymmetric functions
2. An equivalence relation on words
3. Noncommutative P-symmetric functions

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2. An equivalence relation on words
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Natural unit interval orders

Fix a positive integer n. Let $\mathrm{m} = (m_1, \ldots, m_n)$ be a weakly increasing sequence such that $i \leq m_i \leq n$ for each i. Then the *natural unit interval order* $P(\mathrm{m})$ corresponding to m is a poset on $[n] = \{1, 2, \ldots, n\}$ with the ordering $<_P$ given by

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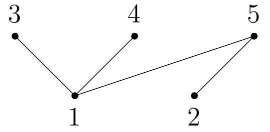
Example

Let $P = P(2, 4, 5, 5, 5)$.



Words on $[n]$

Let P be a poset on $[n]$, and $\mathrm{w} = \mathrm{w}_1\mathrm{w}_2\cdots\mathrm{w}_d$ a word on $[n]$.

- w is of *type* $\mu = (\mu_1, \dots, \mu_n)$ if i appears μ_i times in w. Denote by $W(\mu)$ the set of words of type μ.
- i is a *P-descent* of w if $\mathrm{w}_i >_P \mathrm{w}_{i+1}$, and

$$\mathrm{Des}_P(\mathrm{w}) = \{i \in [d-1] \mid \mathrm{w}_i >_P \mathrm{w}_{i+1}\}.$$

- $(i < j)$ is a *P-inversion pair* of w if w_i and w_j are incomparable in P, and $\mathrm{w}_i > \mathrm{w}_j$ in the natural order on $\mathbb{P}$, and $\mathrm{inv}_P(\mathrm{w})$ denotes the number of P-inversion pairs.

Chromatic symmetric functions

In 1995, Stanley introduced *chromatic symmetric functions*, which are a generalization of chromatic polynomials.
For a poset P,

$$\omega X_P(\mathrm{x}) = \sum_{\mathrm{w} \in W(1^n)} F_{n,\mathrm{Des}_P(\mathrm{w})}(\mathrm{x}).$$

The Stanley–Stembridge conjecture

For a natural unit interval order P, let

$$\omega X_P(\mathrm{x}) = \sum_{\lambda} c_\lambda h_\lambda(\mathrm{x}),$$

then $c_\lambda \geq 0$ for all λ.

Chromatic quasisymmetric functions

In 2016, Shareshian and Wachs introduced a quasisymmetric generalization of a chromatic symmetric function: for a poset P,

$$\omega X_P(\mathrm{x}, q) = \sum_{\mathrm{w} \in W(1^n)} q^{\mathrm{inv}_P(\mathrm{w})} F_{n, \mathrm{Des}_P(\mathrm{w})}(\mathrm{x}).$$

They proved that $\omega X_P(\mathrm{x}, q)$ is also symmetric when P is a natural unit interval order.

Theorem (Gasharov 96', Shareshian–Wachs 16')

For a natural unit interval order P, $\omega X_P(\mathrm{x}, q)$ is Schur positive.

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The refined Stanley–Stembridge conjecture (Shareshian–Wachs 16')

For a natural unit interval order P, let

$$\omega X_P(\mathrm{x}, q) = \sum_{\lambda} c_\lambda(q) h_\lambda(\mathrm{x}),$$

then $c_\lambda(q) \in \mathbb{N}[q]$ *for all* λ*.*

Advantage of the quasisymmetric refinement

Obviously, the refined conjecture is stronger than the original conjecture. On the other hand, the P-inversion statistic gives us a hint for the e-positivity conjecture. Indeed, we can cluster words along with their P inversions, and thus concentrate certain words rather than the whole words.

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In addition, the quasisymmetric refinement gives us a bridge between the chromatic symmetric functions and the cohomology of Hessenberg varieties, and the q-grading on words plays the role of the cohomology degree.

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Question

Can we refine the chromatic quasisymmetric functions $\omega X_P(\mathrm{x}, q)$?

Chromatic quasisymmetric functions

Fix a natural unit interval order P on $[n]$. For $\mu \in \mathbb{N}^n$, define *(multi-)chromatic quasisymmetric function* $\omega X_P(\mathrm{x}, q; \mu)$ to be

$$\omega X_P(\mathrm{x}, q; \mu) = \sum_{\mathrm{w} \in W(\mu)} q^{\mathrm{inv}_P(\mathrm{w})} F_{d, \mathrm{Des}_P(\mathrm{w})}(\mathrm{x}),$$

where $d = \mu_1 + \cdots + \mu_n$.
By definition, $\omega X_P(\mathrm{x}) = \omega X_P(\mathrm{x}, 1; (1^n))$ and $\omega X_P(\mathrm{x}, q) = \omega X_P(\mathrm{x}, q; (1^n))$.

An equivalence relation on words

For $\mu \in \mathbb{N}^n$, let Γ_μ be a graph whose vertex set is $W(\mu)$ and two words w and v are adjacent if and only if (w, v) are of one of the following forms:

(i) $\begin{cases} \mathsf{w} & = \cdots \mathsf{ac} \cdots \\ \mathsf{v} & = \cdots \mathsf{ca} \cdots \end{cases}$ if $\mathsf{a} <_P \mathsf{c}$.

(ii) $\begin{cases} \mathsf{w} & = \cdots \mathsf{acb} \cdots \\ \mathsf{v} & = \cdots \mathsf{bac} \cdots \end{cases}$ if $\mathsf{a} < \mathsf{b} < \mathsf{c}$, $\mathsf{a} \not<_P \mathsf{b} \not<_P \mathsf{c}$, and $\mathsf{a} <_P \mathsf{c}$.

For $\mathsf{w}, \mathsf{v} \in W(\mu)$, we say $\mathsf{w} \sim \mathsf{v}$ if they belong to the same connected component in Γ_μ.

Example

Let $P = P(2,3,3)$ and $\mu = (1,1,2)$. Then $W(\mu)$ consists of 12 words.

1233	1323	3123	1332	3132	3312	
	2133	2313	2331	3213	3231	3321

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$$2133 - 2313 - 2331 \qquad 3213 - 3231 \qquad 3321$$

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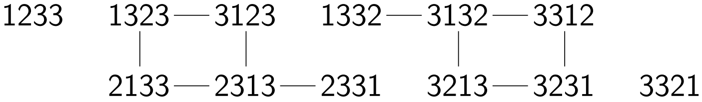

Proposition (H.)

For $\mathrm{w}, \mathrm{v} \in W(\mu)$,

$$\mathrm{w} \sim \mathrm{v} \implies \mathrm{inv}_P(\mathrm{w}) = \mathrm{inv}_P(\mathrm{w}),$$

but the converse does not hold.

Let Γ be a connected component in Γ_μ, and

$$K_\Gamma(\mathrm{x}) = \sum_{\mathrm{w} \in \Gamma} F_{d,\mathrm{Des}_P(\mathrm{w})}(\mathrm{x}).$$

Theorem (H.)

For a connected component Γ, $K_\Gamma(\mathrm{x})$ *is symmetric. In particular,* $\omega X_P(\mathrm{x}, q; \mu)$ *is symmetric.*

A refinement of the refined Stanley–Stembridge conjecture

Conjecture 1 (Stanley–Stembridge 93', Stanley 95')

$\omega X_P(\mathrm{x}, 1; (1^n))$ *is h-positive.*

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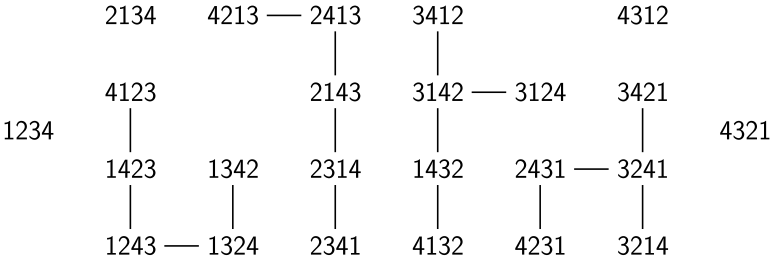
Conjecture 3 (H.)

For each connected component Γ, $K_\Gamma(\mathrm{x})$ *is h-positive. In particular,* $\omega X_P(\mathrm{x}, q; \mu)$ *is h-positive.*

Example

Let $P = P(3,3,4,4)$ and $\mu = (1,1,1,1)$.



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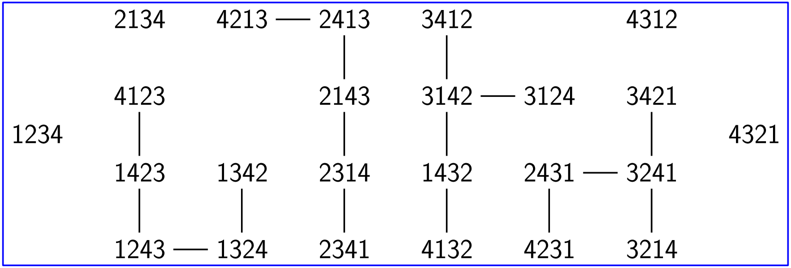

Conjecture 1

Example

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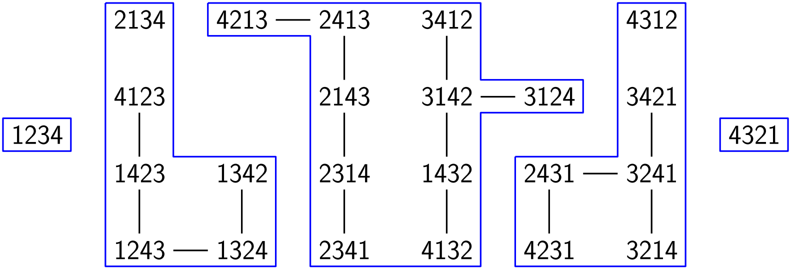
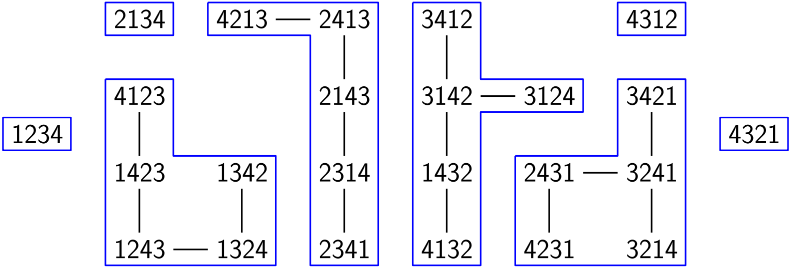


Conjecture 2

Example

Let $P = P(3,3,4,4)$ and $\mu = (1,1,1,1)$.



Conjecture 3

Example

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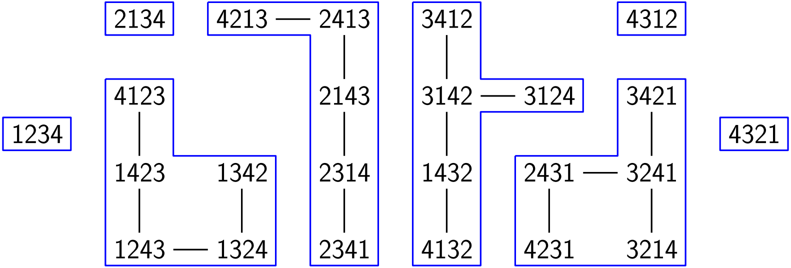

Conjecture 3 $\Longrightarrow$ Conjecture 2 $\Longrightarrow$ Conjecture 1

Natural unit interval orders vs. $(3+1)$-free posets

Natural unit interval orders are characterized by $(3+1)$- and $(2+2)$-freeness.

$$\{\text{natural unit interval orders}\} \subsetneq \{(3+1)\text{-free posets}\}$$

The original Stanley–Stembridge conjecture is about $(3+1)$-free posets, not natural unit interval orders.
The graph structure on $W(\mu)$ and the equivalence relation can be defined for $(3+1)$-free posets. Hence Conjecture 3 can be extended to $(3+1)$-free posets. See Blasiak–Eriksson–Pylyavskyy–Siegl 22'.

Noncommutative P-symmetric functions

Let $\mathcal{U} = \mathbb{Z}\langle u_1, \ldots, u_n \rangle$ be the free associative $\mathbb{Z}$-algebra generated by $\{u_1, \ldots, u_n\}$. For simplicity we often write $u_{\mathrm{w}} = u_{\mathrm{w}_1} u_{\mathrm{w}_2} \cdots u_{\mathrm{w}_d}$ for a word $\mathrm{w} = \mathrm{w}_1 \mathrm{w}_2 \cdots \mathrm{w}_d$ on $[n]$.
Let $\mathcal{I}_P$ be the 2-sided ideal of $\mathcal{U}$ generated by the following elements:

$$
\begin{aligned}
&u_a u_c - u_c u_a && (a <_P c),\\
&u_a u_c u_b - u_b u_a u_c && (a < b < c,\ a \not<_P b \not<_P c \text{ and } a <_P c).
\end{aligned}
$$

Definition

For $k \geq 1$, the *noncommutative P-elementary symmetric function* $\mathfrak{e}_k(\mathrm{u})$ is

$$\mathfrak{e}_k(\mathrm{u}) = \sum_{i_1 >_P i_2 >_P \cdots >_P i_k} u_{i_1} u_{i_2} \cdots u_{i_k} \in \mathcal{U}.$$

Example

Let $P = P(2, 4, 5, 5, 5)$.
Then $\mathfrak{e}_2(\mathrm{u}) = u_3 u_1 + u_4 u_1 + u_5 u_1 + u_5 u_2$, and $\mathfrak{e}_3(\mathrm{u}) = 0$.

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Definition

For $k \geq 1$, define the *noncommutative P-complete homogeneous symmetric function* $\mathfrak{h}_k(\mathrm{u})$ by

$$\mathfrak{h}_k(\mathrm{u}) = \sum_{i_1 \not>_P i_2 \not>_P \cdots \not>_P i_k} u_{i_1} u_{i_2} \cdots u_{i_k} \in \mathcal{U}.$$

Theorem (H.)

For $k, \ell \geq 1$, $\mathfrak{e}_k(\mathrm{u})$ and $\mathfrak{e}_\ell(\mathrm{u})$ commute with each other modulo $\mathcal{I}_P$, that is,

$$\mathfrak{e}_k(\mathrm{u})\mathfrak{e}_\ell(\mathrm{u}) \equiv \mathfrak{e}_\ell(\mathrm{u})\mathfrak{e}_k(\mathrm{u}) \mod \mathcal{I}_P.$$

The noncommutative P-Cauchy product

Recall the Cauchy product:

$$C(\mathrm{x},\mathrm{y}) = \prod_{i,j} \frac{1}{1 - x_i y_j} = \prod_{j\geq 1} \sum_{\ell\geq 0} x_j^{\ell} h_{\ell}(\mathrm{y}).$$

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Definition (H.)

The *noncommutative P-Cauchy product* $\Omega(\mathrm{x},\mathrm{u})$ is

$$\Omega(\mathrm{x},\mathrm{u}) = \prod_{j\geq 1} \sum_{\ell \geq 0} x_j^{\ell} \mathfrak{h}_{\ell}(\mathrm{u}) \in \mathcal{U}[[\mathrm{x}]].$$

Observing the definition of $\Omega(\mathrm{x},\mathrm{u})$ carefully, we have

$$\Omega(\mathrm{x},\mathrm{u}) = \sum_{\mathrm{w}} F_{d,\mathrm{Des}_P(\mathrm{w})}(\mathrm{x}) u_{\mathrm{w}},$$

where w ranges over all words on $[n]$ and d is the length of w.

Let

$$\mathcal{U}^* = \mathbb{Z}\langle \mathrm{w} \mid \mathrm{w} \text{ is a word on } [n] \rangle \quad \text{and} \quad \mathcal{U}_q^* := \mathbb{Z}[q] \otimes \mathcal{U}^*,$$

and define a bilinear pairing between $\mathcal{U}$ and $\mathcal{U}^*$ (and $\mathcal{U}_q^*$) by

$$\langle u_{\mathrm{w}}, \mathrm{v} \rangle := \delta_{\mathrm{w},\mathrm{v}} \text{ for words w and v.}$$

Let

$$\gamma_\Gamma = \sum_{\mathrm{w} \in \Gamma} \mathrm{w} \in \mathcal{U}^* \quad \text{and} \quad \gamma_\mu = \sum_{\mathrm{w} \in W(\mu)} q^{\mathrm{inv}_P(\mathrm{w})} \mathrm{w} \in \mathcal{U}_q^*.$$

Then by definition,

$$K_\Gamma(\mathrm{x}) = \langle \Omega(\mathrm{x}, \mathrm{u}), \gamma_\Gamma \rangle \quad \text{and} \quad \omega X_P(\mathrm{x}, q; \mu) = \langle \Omega(\mathrm{x}, \mathrm{u}), \gamma_\mu \rangle.$$

Theorem (H.)

Suppose that

$$\Omega(\mathrm{x},\mathrm{u}) \equiv \sum_{\lambda} g_\lambda(\mathrm{x})\mathfrak{f}_\lambda(\mathrm{u}) \mod \mathcal{I}_p[[\mathrm{x}]].$$

Let $K_\Gamma(\mathrm{x}) = \sum_\lambda r_{\lambda,\Gamma} g_\lambda(\mathrm{x})$ *and* $\omega X_P(\mathrm{x}, q; \mu) = \sum_\lambda r_\lambda(q) g_\lambda(\mathrm{x})$*. Then for any* λ*,*

$$r_{\lambda,\Gamma} = \langle \mathfrak{f}_\lambda(\mathrm{u}), \gamma_\Gamma \rangle \quad \textit{and} \quad r_\lambda(q) = \langle \mathfrak{f}_\lambda(\mathrm{u}), \gamma_\mu \rangle.$$

In particular, if $\mathfrak{f}_\lambda(\mathrm{u})$ *has a positive monomial expression modulo* $\mathcal{I}_P$ *for all* λ*, then* $K_\Gamma(\mathrm{x})$ *and* $\omega X_P(\mathrm{x}, q; \mu)$ *are* g*-positive.*

Noncommutative P-Schur functions

Recall that for a partition λ,

$$s_\lambda(\mathrm{x}) = \det(e_{\lambda'_i - i + j}(\mathrm{x}))_{i,j} = \sum_{\sigma \in \mathfrak{S}_m} \mathrm{sgn}(\sigma) e_{\lambda'_1 - 1 + \sigma_1}(\mathrm{x}) \cdots e_{\lambda'_m - m + \sigma_m}(\mathrm{x}),$$

where λ' is the conjugate of λ and $m = \lambda_1$.

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where λ' is the conjugate of λ and $m = \lambda_1$.

Definition (H.)

For a partition λ, the *noncommutative P-Schur function* $\mathfrak{J}_\lambda(\mathrm{u})$ is

$$\mathfrak{J}_\lambda(\mathrm{u}) := \sum_{\sigma \in \mathfrak{S}_m} \mathrm{sgn}(\sigma) \mathfrak{e}_{\lambda'_1 - 1 + \sigma_1}(\mathrm{u}) \cdots \mathfrak{e}_{\lambda'_m - m + \sigma_m}(\mathrm{u}).$$

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Proposition (H.)

We have

$$\Omega(\mathrm{x},\mathrm{u}) \equiv \sum_\lambda s_\lambda(\mathrm{x})\mathfrak{J}_\lambda(\mathrm{u}) \mod \mathcal{I}_P[[\mathrm{x}]].$$

Definition

For a partition λ, a *semistandard P-tableau* of shape λ is a filling of Young diagram of shape λ with $[n]$ satisfying that

(i) each row is non-P-decreasing from left to right, and

(ii) each column is P-increasing from top to bottom.

- T is of *type* $\mu = (\mu_1, \dots, \mu_n)$ if each $i \in [n]$ appears μ_i times in T.
- $\mathrm{w}(T)$ = the column reading word of T.
- $\mathcal{T}_P(\lambda)$ = the set of all semistandard P-tableaux of shape λ.

Theorem (H.)

For a partition λ, we have

$$\mathfrak{J}_\lambda(\mathrm{u}) \equiv \sum_{T \in \mathcal{T}_P(\lambda)} u_{\mathrm{w}(T)} \mod \mathcal{I}_P.$$

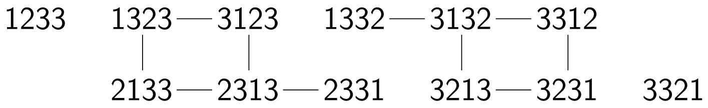
Consequently,

$$\omega X_P(\mathrm{x}, q; \mu) = \sum_T q^{\mathrm{inv}_P(\mathrm{w}(T))} s_{\mathrm{sh}(T)}(\mathrm{x}),$$

where T ranges over all semistandard P-tableaux of type μ and $\mathrm{sh}(T)$ denotes the shape of T.

Example

Let $P = P(2,3,3)$ and $\mu = (1,1,2)$. Then



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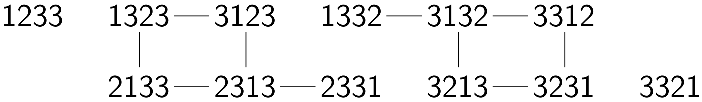


1	2	3	3

1	3	2	3

1	3	3	2

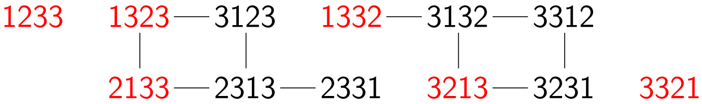
2	1	3	3

3	2	1	3

3	3	2	1

Example

Let $P = P(2,3,3)$ and $\mu = (1,1,2)$. Then



$s_4(x)$ $2s_4(x)$ $2s_4(x)$ $s_4(x)$

1	2	3	3
2	1	3	3

1	3	2	3
3	2	1	3

1	3	3	2
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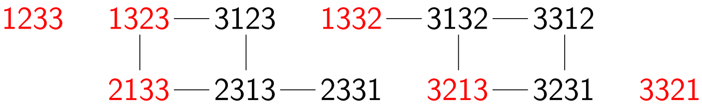

$s_4(x)$ $\quad$ $2s_4(x)$ $\quad$ $2s_4(x)$ $\quad$ $s_4(x)$

1	2	3
3		

1	3	2
3		

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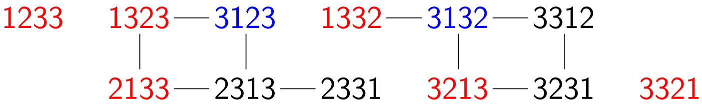

$s_4(\mathrm{x})$ $\quad$ $2s_4(\mathrm{x}) + s_{3,1}(\mathrm{x})$ $\quad$ $2s_4(\mathrm{x}) + s_{3,1}(\mathrm{x})$ $\quad$ $s_4(\mathrm{x})$

1	2	3
3		

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1	2	3
3		

1	3	2
3		

$$\omega X_P(\mathrm{x}, q; \mu) = (q^3 + 2q^2 + 2q + 1)s_4(\mathrm{x}) + (q^2 + q)s_{3,1}(\mathrm{x}).$$

Noncommutative P-monomial symmetric functions

Let $(N_{\lambda,\mu})$ be the transition matrix between monomial symmetric functions $m_\lambda(\mathrm{x})$ and elementary symmetric functions $e_\mu(\mathrm{x})$, i.e.,

$$m_\lambda(\mathrm{x}) = \sum_\mu N_{\lambda,\mu} e_\mu(\mathrm{x}).$$

Definition (H.)

For a partition λ, the *noncommutative P-monomial symmetric function* $\mathfrak{m}_\lambda(\mathrm{u})$ is

$$\mathfrak{m}_\lambda(\mathrm{u}) := \sum_\mu N_{\lambda,\mu} \mathfrak{e}_\mu(\mathrm{u}).$$

Proposition (H.)

We have

$$\Omega(\mathrm{x},\mathrm{u}) \equiv \sum_{\lambda} h_\lambda(\mathrm{x})\mathfrak{m}_\lambda(\mathrm{u}) \mod \mathcal{I}_P[[\mathrm{x}]].$$

Therefore, finding a positive monomial expression of $\mathfrak{m}_\lambda(\mathrm{u})$ is equivalent to finding a positive combinatorial interpretation of the coefficients of the h-expansion of $K_\Gamma(\mathrm{x})$, which proves Conjecture 3.

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Theorem (H.)

For a partition λ of hook shape or 2-column shape, $\mathfrak{m}_\lambda(\mathrm{u})$ has a positive monomial expression modulo $\mathcal{I}_P$.
In particular, let $K_\Gamma(\mathrm{x}) = \sum_\lambda c_{\lambda,\Gamma} h_\lambda(\mathrm{x})$, then $c_{\lambda,\Gamma} \geq 0$ if λ is of hook shape or 2-column shape.

Further directions

(1) Equivalence relation on the Hessenberg varieties.
The q-grading (the P-inversion statistic) on the chromatic quasisymmetric functions has a geometric meaning: the cohomology degree of the Hessenberg variety. The equivalence relation refines the statistic, and the equivalence classes define Schur positive symmetric functions. Is there a geometric meaning of the equivalence classes?

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(2) Equivalence relation on unicellular LLT polynomials.
There is a plethystic formula between chromatic quasisymmetric functions and unicellular LLT polynomials:

$$\mathrm{LLT}_P(\mathrm{x}, q) = (q-1)^n X_P[\mathrm{x}/(q-1); q].$$

Is there a similar equivalence relation on monomials of LLT polynomials?

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Thank you!