Interpolating between ordinary and bumpless pipe dreams with hybrid pipe dreams

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Joint work with Allen Knutson

July 17, 2023

Outline

1. Schubert Polynomials
2. Pipe dreams compute Schubert polynomials
3. Bumpless pipe dreams compute Schubert polynomials
4. Hybrid pipe dreams mix pipe dreams and bumpless pipe dreams and compute Schubert polynomials
5. Bijecting pipe dreams and bumpless pipe dreams with and without hybrid pipe dreams

Schubert Polynomials

1. For $\pi \in S_n$, $S_\pi(\mathbf{x}, \mathbf{y})$ is the double Schubert polynomial for π and $S_\pi(\mathbf{x}, 0)$ is the (single) Schubert polynomial for π
2. Schubert polynomials have positive integer coefficients
3. $\{S_\pi\}_{\pi \in S_\infty}$ is a $\mathbb{Z}$-basis for $\mathbb{Z}[x_1, x_2, \ldots]$
4. The Schubert basis includes the Schur polynomials which give a basis for symmetric polynomials
5. Schubert polynomials represent classes of Schubert varieties in the cohomology of the flag variety
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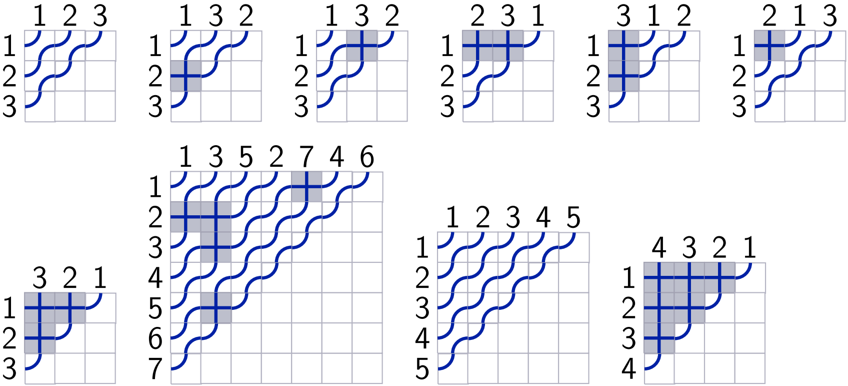
Pipe dreams by example



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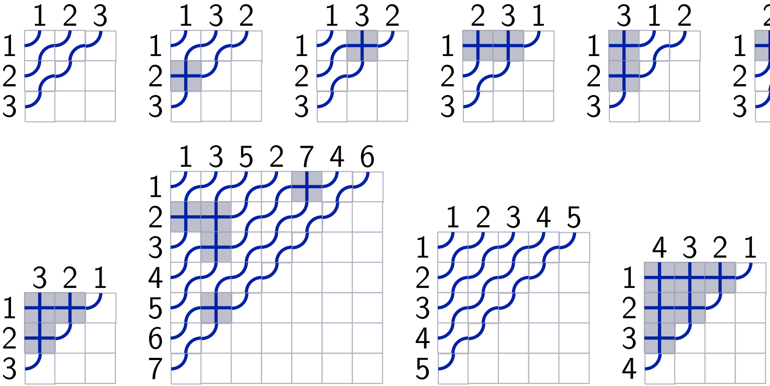

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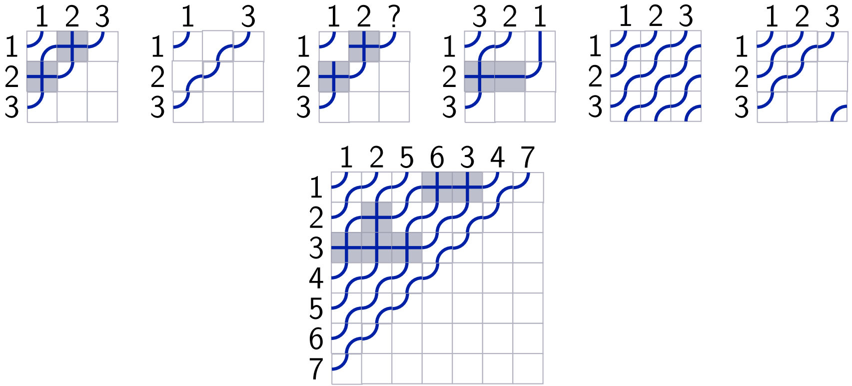
Not pipe dreams



Pipe dreams

Definition (Bergeron-Billey 1993)

An (ordinary) **pipe dream** for $\pi \in S_n$ is a filling of a square with

so two 'pipes' never cross twice, each pipe connects some i on top to i on the left when top edges are labeled $\pi^{-1}(1), \ldots, \pi^{-1}(n)$ and left edges are labeled $1, \ldots, n$ from top to bottom.

Basic properties

1. , and provably never appear
2. the antidiagonal is all
3. the SE triangle is all
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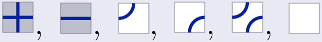 but not ,

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Computing (double) Schubert polynomials

Definition

The set of pipe dreams for π is $\mathrm{PD}(\pi)$.
Numbering rows and columns $1, \ldots, n$ from top to bottom and left to right, define

weighty$(D) = \{(i, j)$: tile in ith row & jth column of D is  or $\}$.

Theorem (Fomin-Kirillov 1996)

The double Schubert polynomial of π can be computed as

$$\mathcal{S}_\pi(\mathbf{x}, \mathbf{y}) = \sum_{D \in \mathrm{PD}(\pi)} \prod_{(i,j) \in \mathrm{weighty}(D)} (x_i - y_j)$$

$\mathcal{S}_\pi(\mathbf{x}, 0) = \mathcal{S}_{\pi^{-1}}(0, -\mathbf{x})$ is the Schubert polynomial of π.

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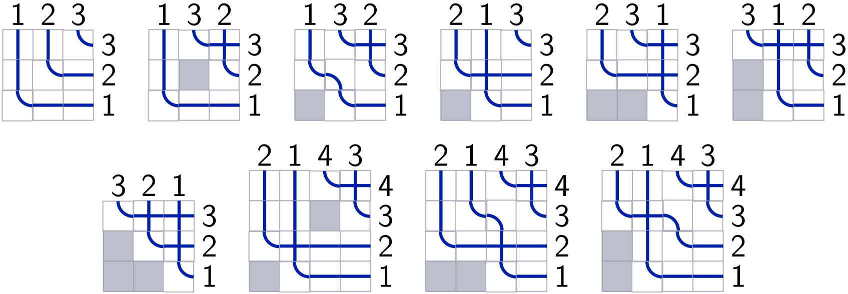
(Upside down) Bumpless pipe dreams by example



Bumpless pipe dreams

Definition (Lam-Lee-Shimozono 2018)

An (upside down) **bumpless pipe dream** for π is a filling of an $n \times n$ square with tiles

 but not ,

so that two pipes never cross twice and pipes connect i on top to i on the right when the top edges of the square are labeled $\pi^{-1}(1), \ldots, \pi^{-1}(n)$ and the right edges are labeled $1, \ldots, n$ from bottom to top.

Bumpless pipe dreams and Schubert polynomials

Definition

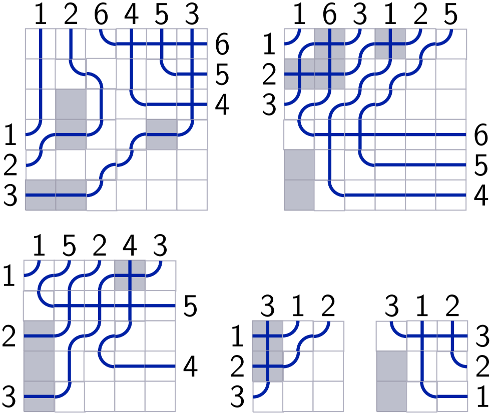
Write $\mathrm{BPD}(\pi)$ for the set of bumpless pipe dreams for π.

$$\mathrm{weighty}(D) := \{(i,j) : \text{the tile in } i\text{th row \& } j\text{th column of } D \text{ is } \square\}.$$

Theorem (Lam-Lee-Shimozono 2018)

$$S_\pi(\mathbf{x},\mathbf{y}) = \sum_{D \in \mathrm{BPD}(\pi)} \prod_{(i,j)\in \mathrm{weighty}(D)} (x_i - y_j)$$

Hybrid pipe dreams by example



Hybrid pipe dreams defined

The ordinary tiles are

and the bumpless tiles are

Definition (Knutson)

A **hybrid pipe dream** is a filling of a square with ordinary and bumpless tiles such that

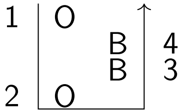
1. each row is either ordinary or bumpless
2. pipes connect top edges to left and right edges
3. no two pipes cross twice

Hybrid types

Definition

A hybrid pipe dream is of **type** $\tau = \tau_1\tau_2 \ldots \tau_n \in \{O, B\}^n$ if (its ith physical row is ordinary if $\tau_i = O$ and bumpless if $\tau_i = B$).

Labels are left of ordinary rows and right of bumpless rows. Ordinary rows labeled $1, \ldots, k$ from the top. Bumpless rows labeled $k \ldots, n$ from the bottom.



Definition

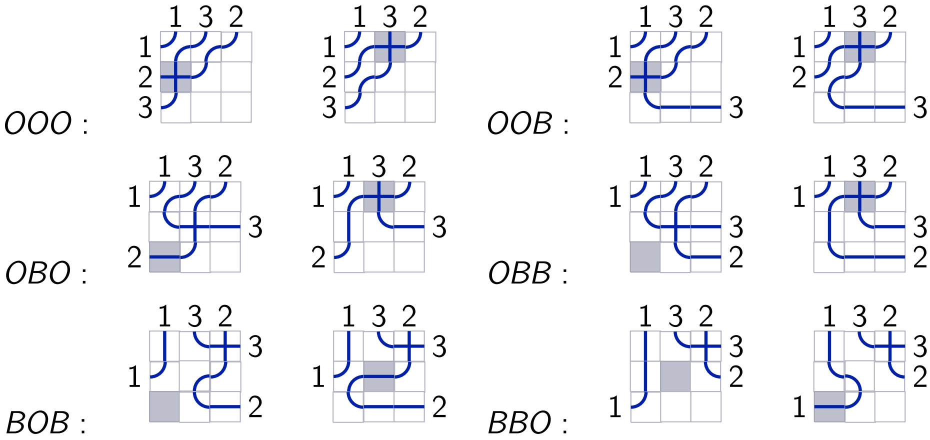
A hybrid pipe dream **for** π has labels $\pi^{-1}(1), \ldots, \pi^{-1}(n)$ across the top so pipes connect rows and columns with the same label.

Definition

$\mathrm{HPD}(\pi, \tau)$ is the set of hybrid pipe dreams for π with type τ.

More hybrid examples

Pipe dreams are the hybrid pipe dreams of type O^n, bumpless pipe dreams are the hybrid pipe dreams of type B^n.
Hybrid pipe dreams for $\pi = 132$ and some choices of τ.



Schubert polynomials from hybrid pipe dreams

 and  are the two weighty ordinary tiles.
 is the only weighty bumpless tile.

Definition

$\mathrm{weighty}(D) = \{(i, j) :$
tile in the row labeled i and jth physical column of D is weighty$\}$.

Theorem (Knutson)

For any $\tau \in \{O, B\}^n$,

$$S_\pi = \sum_{D \in \mathrm{HPD}(\pi,\tau)} \prod_{(i,j) \in \mathrm{weighty}(D)} (x_i - y_j)$$

Single Schubert polynomials are slightly simpler

Definition

$$\text{mon}_x(D) = \prod_{(i,j)\in\text{weighty}(D)} x_i, \text{ and } \qquad \text{mon}_y(D) = \prod_{(i,j)\in\text{weighty}(D)} -y_j.$$

Lemma

$$S_\pi(\mathbf{x}, 0) = \sum_{D\in\text{HPD}(\pi,\tau)} \text{mon}_x(D) = \sum_{D\in\text{HPD}(\pi^{-1},\tau)} \text{mon}_y(D)(-x_1, \dots, -x_n)$$

If D^T is the transpose of a pipe dream (or the transpose about the antidiagonal of a bumpless pipe dream) then

$$\text{mon}_x(D^T)(-y_1, \dots, -y_n) = \text{mon}_y(D), \text{ and}$$

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Connecting ordinary and bumpless pipe dreams

Theorem (Monk's rule)

Let π be a permutation and α be a positive integer, then

$$x_\alpha \mathfrak{S}_\pi(\mathbf{x},0) + \sum_{s<\alpha, \pi t_{s,\alpha} \gtrdot \pi} \mathfrak{S}_{\pi t_{s,\alpha}}(\mathbf{x},0) = \sum_{l>\alpha, \pi t_{\alpha,l} \gtrdot \pi} \mathfrak{S}_{\pi t_{\alpha,l}}(\mathbf{x},0)$$

- Huang 2020 gave a bijective proof of Monk's rule with BPDs
- Gao-Huang 2021 gave a bijective proof of this Monk's rule with PDs (based on Billey-Holroyd-Young 2019)

Theorem (Gao-Huang 2021)

There is a (non-equivariant) weight preserving bijection between pipe dreams and bumpless pipe dreams ϕ_{GH} which is canonical in the sense that it commutes with Monk's rule.

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Special cases of a more general result

Hybrid pipe dreams prove the equivalence of formulæ for double Schubert polynomials

Theorem (Knutson-U)

For $\tau_1, \tau_2 \in \{O, B\}^n$, and any $\pi \in S_n$, there exists a bijection $\phi_x : \mathrm{HPD}(\pi, \tau_1) \to \mathrm{HPD}(\pi, \tau_2)$ such that $\mathrm{mon}_x(\phi_x(D)) = \mathrm{mon}_x(D)$

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Plan for constructing hybrid bijections

Part 1: switch the last row

Define $\phi_y : \mathrm{HPD}(\pi, \tau) \to \mathrm{HPD}(\pi, \tau')$ for

$$\tau = \tau_1\tau_2 \ldots \tau_{n-1}O, \qquad \tau' = \tau_1\tau_2 \ldots \tau_{n-1}B.$$

Only change the last row of the hybrid pipe dream.

Part 2: swap adjacent rows

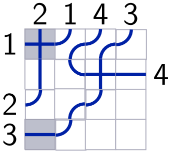
Define $\phi_y : \mathrm{HPD}(\pi, \tau) \to \mathrm{HPD}(\pi, \tau')$ for

$$\tau = \tau_1\tau_2 \ldots \tau_{k-1}OB\tau_k \ldots \tau_n \qquad \tau' = \tau_1\tau_2 \ldots \tau_{k-1}BO\tau_k \ldots \tau_n.$$

Modify only the physical kth, $(k+1)$st rows.

E.g. OOO $\xrightarrow{1}$ OOB $\xrightarrow{2}$ OBO $\xrightarrow{2}$ BOO $\xrightarrow{1}$ BOB $\xrightarrow{2}$ BBO $\xrightarrow{1}$ BBB

$\Leftrightarrow$



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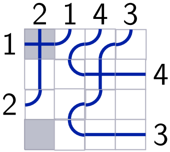
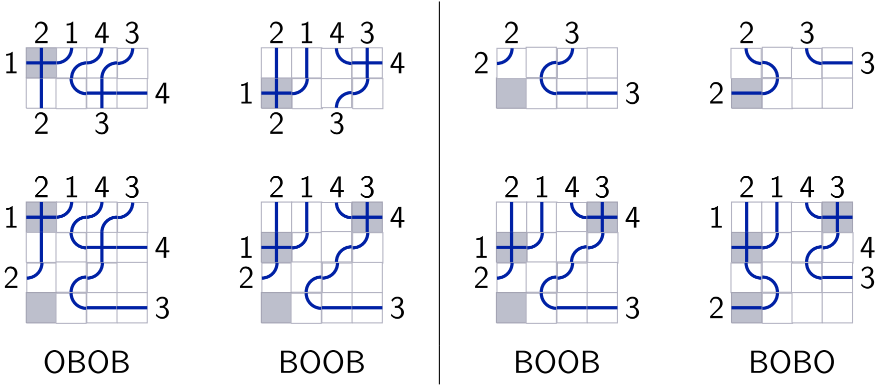
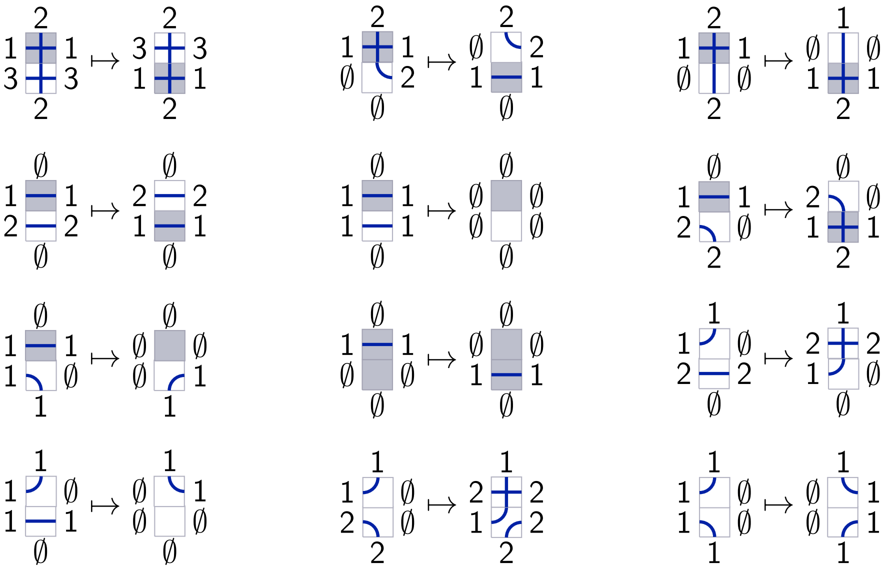


Switching two adjacent rows



The row swapping bijection ϕ_y (Page 1 of 2)



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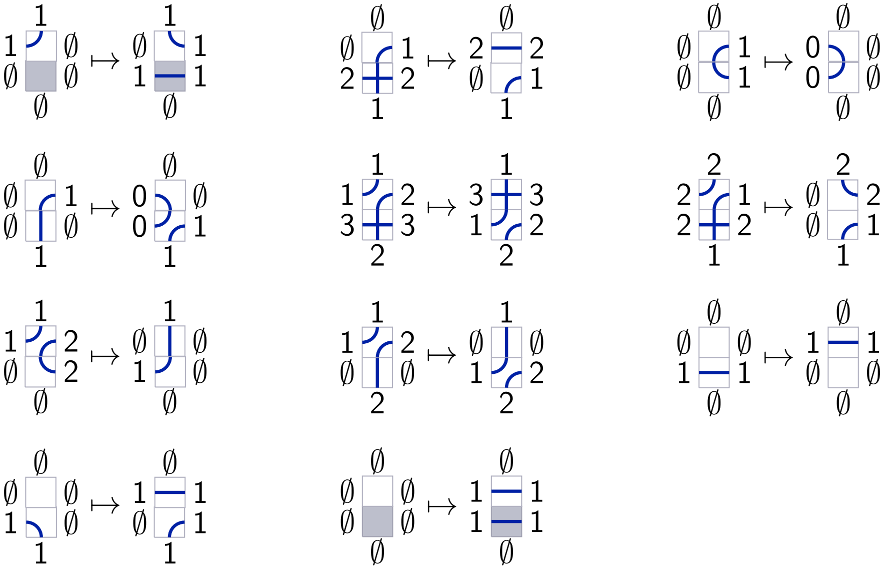

A third approach to an x-weight preserving bijection from ordinary to bumpless pipe dreams

Suppose $\phi_y : \mathrm{HPD}(\pi, O^n) \to \mathrm{HPD}(\pi, B^n)$ is a bijection and commutes with mon_y .

$$\begin{aligned}\mathrm{mon}_x\left(\left(\phi_y\left(D^\mathsf{T}\right)\right)^\mathsf{T}\right) &= \mathrm{mon}_y\left(\phi_y\left(D^\mathsf{T}\right)\right)\\ &= \mathrm{mon}_y\left(D^\mathsf{T}\right)\\ &= \mathrm{mon}_x(D)\end{aligned}$$

Therefore $D \mapsto \left(\phi_y\left(D^\mathsf{T}\right)\right)^\mathsf{T}$ gives a bijection between ordinary pipe dreams and bumpless pipe dreams that commutes with mon_x

Work in progress (Knutson-U)

$\left(\phi_y\left(D^\mathsf{T}\right)\right)^\mathsf{T} = \phi_x(D) = \phi_{GH}(D)$

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Decorated hybrid pipe dreams

Definition (Huang 2020)

$$\widetilde{\mathrm{HPD}}(\pi, \tau) = \{(D, f) : D \in \mathrm{HPD}(\pi, \tau), f : \mathrm{weighty}(D) \to \{x, -y\}\}$$

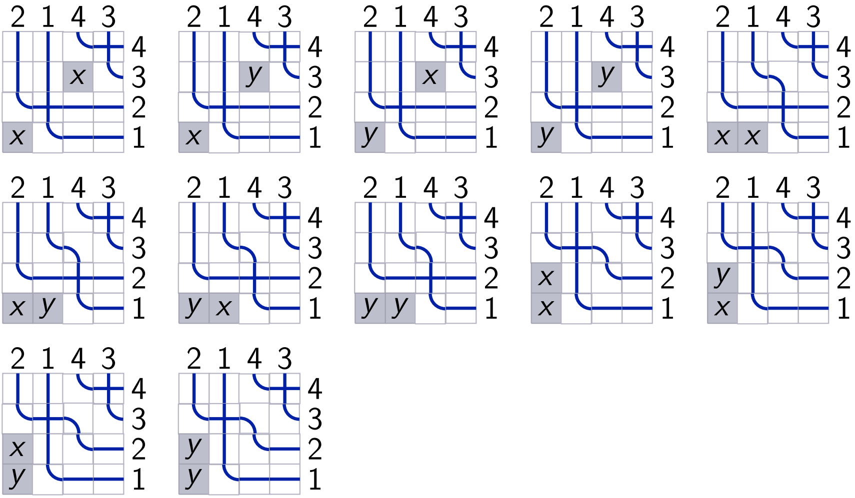
$$\mathrm{mon}(D, f) = \prod_{\substack{(i,j) \in \mathrm{weighty}(D) \\ f(i,j)=x}} x_i \prod_{\substack{(i,j) \in \mathrm{weighty}(D) \\ f(i,j)=-y}} -y_j$$

$$S_\pi(\mathbf{x}, \mathbf{y}) = \sum_{(D,f) \in \widetilde{\mathrm{HPD}}(\pi,\tau)} \mathrm{mon}(D, f)$$

Theorem (Knutson-U)

For $\tau_1, \tau_2 \in \{O, B\}$, and $\pi \in S_n$ there exists a bijection $\phi : \widetilde{\mathrm{HPD}}(\pi, \tau_1) \to \widetilde{\mathrm{HPD}}(\pi, \tau_2)$ such that $\mathrm{mon}(\phi(D, f)) = \mathrm{mon}(D, f)$

Examples of decorated bumpless pipe dreams



There is lots to do!

Further questions

- Describe (equivariant) Monk's rule maps on decorated hybrid pipe dreams
- Describe the bijection between decorated PDs and decorated BPDs without using hybrid pipe dreams

Acknowledgements

We thank Daoji Huang and Paul Zinn-Justin for many pertinent conversations.
We are grateful to Adam Gregory/Zachary Hamaker/Hugh Dennin for writing LaTeX macros used in this talk.
We are extremely grateful to Tianyi Yu, who was instrumental in finding the (mostly finished) proof that $\phi_y(D^\mathsf{T})^\mathsf{T} = \phi_{GH}(D)$

Bonus slides

There is no equivariant weight preserving bijection

(Impossible) Goal

Find a weight preserving bijection between any two types of hybrids.
Hope: Factoring the PD to BPD bijection through HPDs is insightful

Ordinary and bumpless pipe dreams for $\pi = 132$.

$OOO:$

$(x_2 - y_1)$ $(x_1 - y_2)$

$BBB:$

$(x_2 - y_2)$ $(x_1 - y_1)$

$$S_{132} = (x_2 - y_1) + (x_1 - y_2) = (x_2 - y_2) + (x_1 - y_1) = x_1 + x_2 - y_1 - y_2$$

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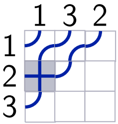

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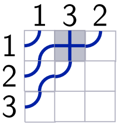

$(x_1 - y_2)$

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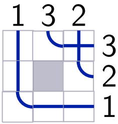

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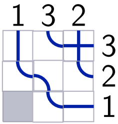

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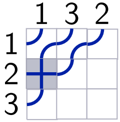
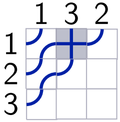
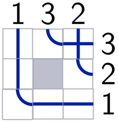
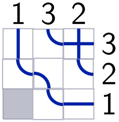
There are non-equivariant weight preserving bijections

Ordinary and bumpless pipe dreams for $\pi = 132$.

OOO :			BBB :		
weight	$(x_2 - y_1)$	$(x_1 - y_2)$		$(x_2 - y_2)$	$(x_1 - y_1)$
$\mathrm{mon}_x(D)$	x_2	x_1		x_2	x_1
$\mathrm{mon}_y(D)$	$-y_1$	$-y_2$		$-y_2$	$-y_1$

$$S_{132}(x_1, x_2, 0, 0) \qquad = x_2 + x_1 \qquad = x_2 + x_1$$

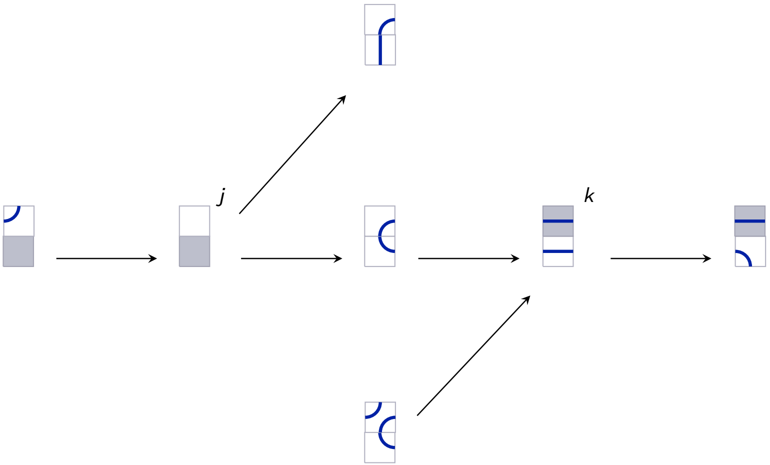
$$S_{132}(0, 0, y_1, y_2) \qquad = (-y_1) + (-y_2) \qquad = (-y_2) + (-y_1)$$

Getting a weight preserving bijection for decorated hybrid pipe dreams

Plan:

- Break up a strip of n dominos into a strip of k dominos and a strip of $(n-k)$ dominos
- Understand the bijection on the irreducible pieces

Irreducible chunks



Irreducible chunks 2