# Pop, Crackle, Snap (and Pow): Some Facets of Shards

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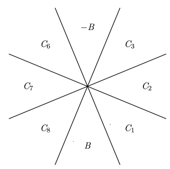
A hyperplane $H$ is a linear codimension-1 subspace of $\mathbb{R}^n$. A central hyperplane arrangement $\mathcal{H}$ is a finite collection of hyperplanes.

Central $\supset$ Simplicial $\supset$ Reflection

Write $\mathcal{R}$ for the set of connected components (the regions) of the complement $\mathbb{R}^n \setminus \mathcal{H}$. Fix $B$ a base region in $\mathcal{R}$.

## Edelman's Regions

For $C \in \mathcal{R}$, define $\mathrm{inv}_{\mathcal{H},B}(C)$ to be the hyperplanes in $\mathcal{H}$ separating $B$ from $C$.



The map $C \mapsto \mathrm{inv}_{\mathcal{H},B}(C) \subseteq \mathcal{H}$ is injective.

The poset of regions $\mathrm{Weak}(\mathcal{H}, B)$ has elements $\mathcal{R}$ and relations

$$C \leq D \text{ iff } \mathrm{inv}_{\mathcal{H},B}(C) \subseteq \mathrm{inv}_{\mathcal{H},B}(D).$$

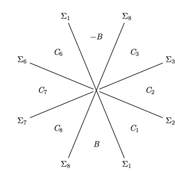
$\mathrm{Weak}(\mathcal{H}, B)$ is a lattice for every $B \in \mathcal{R}$ iff $\mathcal{H}$ is simplicial.

The pop-stack sorting operator $\mathsf{Pop} : \mathcal{R} \to \mathcal{R}$ is $\mathsf{Pop}(C) := \bigwedge_{D \lessdot C} D.$


## Reading's Shards

Reading cut the hyperplanes in an arrangement $\mathcal{H}$ into pieces called shards. Write $\mathrm{inv}_{\mathrm{III},B}(C)$ to be the shards in $\mathcal{H}$ separating $\mathsf{Pop}(C)$ from $C$.

The map $C \mapsto \mathrm{inv}_{\mathrm{III},B}(C) \subseteq \mathrm{III}$ is injective.



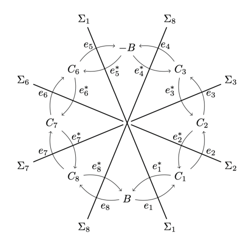
The shard intersection order $\mathrm{Shard}(\mathcal{H}, B)$ has elements $\mathcal{R}$ and relations

$$C \preceq D \text{ iff } \mathrm{inv}_{\mathrm{III},B}(C) \subseteq \mathrm{inv}_{\mathrm{III},B}(D).$$

## Salvetti's Loops

The Salvetti complex $\mathrm{Sal}(\mathcal{H})$ is defined by gluing together oriented dual zonotopes for $\mathcal{H}$ along compatible faces—one zonotope for each choice of base region $B$, oriented from $B$ to $-B$. Write $\mathbb{C}^n \setminus \mathcal{H}_{\mathbb{C}}$ for the complexified hyperplane complement of $\mathcal{H}$.

**Theorem (Salvetti)** $\pi_1(\mathrm{Sal}(\mathcal{H}), B) = \pi_1(\mathbb{C}^n \setminus \mathcal{H}_{\mathbb{C}}, x_B)$.



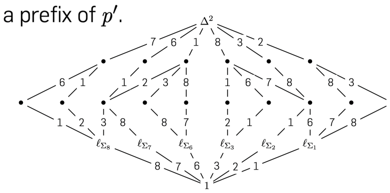
If $C \xrightarrow{e} C'$ is a cover in $\mathrm{Weak}(\mathcal{H}, B)$, define a loop $\ell_e \in \pi_1(\mathrm{Sal}(\mathcal{H}), B)$ by $\ell_e := \mathrm{gal}(B, C) \cdot ee^* \cdot \mathrm{gal}(B, C)^{-1} \in \pi_1(\mathrm{Sal}(\mathcal{H}), B)$.

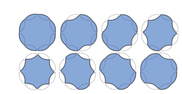
$\pi_1(\mathrm{Sal}(\mathcal{H}), B)$ is generated by the loops $\mathcal{L}_{\mathrm{edge}}$, the set of all such $\ell_e$. One family of relations for each 2-cell.

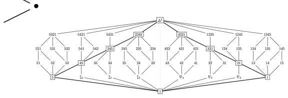
> For a real central arrangement $\mathcal{H}$, $\ell_e \simeq \ell_f$ iff $\Sigma(e) = \Sigma(f)$.

The pure shard monoid $\mathbf{P}^+(\mathcal{H}, B)$ is generated by $\mathcal{L}_{\mathrm{III}}$. $\mathbf{P}^+(\mathcal{H}, B)$ is ordered by $p \leq p'$ if $p$ is a prefix of $p'$.
The full twist $\Delta^2$ lies in the center of $\pi_1(\mathrm{Sal}(\mathcal{H}), B)$.

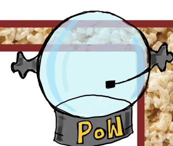
> Claim: the interval $[1, \Delta^2]_{\mathbf{P}^+}$ is an analogue of $\mathrm{Weak}(\mathcal{H}, B)$ and $\mathrm{Shard}(\mathcal{H}, B)$.







## Pow: Weak Embedding




Fix $\mathcal{H}$ central. For $C \in \mathcal{R}$ and a positive minimal gallery

$$B = C_0 \xrightarrow{e_1} C_1 \xrightarrow{e_2} \cdots \xrightarrow{e_{k-1}} C_{k-1} \xrightarrow{e_k} C_k = C,$$

define $\mathsf{Pow}(C) := \ell_{\Sigma(e_k)} \ell_{\Sigma(e_{k-1})} \cdots \ell_{\Sigma(e_1)}$.

> Pow is a poset embedding of $\mathrm{Weak}(\mathcal{H}, B)$ in $[1, \Delta^2]_{\mathbf{P}^+}$.

## Crackle: Shard Embedding

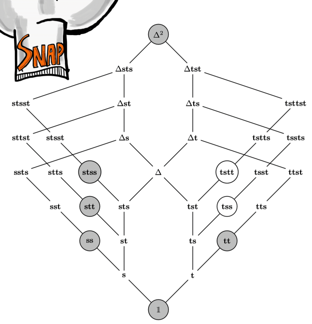

Fix $\mathcal{H}$ simplicial. For $C \in \mathcal{R}$ and a positive minimal gallery

$$\mathsf{Pop}(C) = E_0 \xrightarrow{e_1} E_1 \xrightarrow{e_2} \cdots \xrightarrow{e_{k-1}} E_{k-1} \xrightarrow{e_k} E_k = C$$

define $\mathsf{Crackle}(C) := \ell_{\Sigma(e_k)} \ell_{\Sigma(e_{k-1})} \cdots \ell_{\Sigma(e_1)}$.

> Crackle is a poset embedding of $\mathrm{Shard}(\mathcal{H}, B)$ into the interval $[1, \Delta^2]_{\mathbf{P}^+}$.

## Snap = Crackle · Pop

Fix $\mathcal{H}$ a reflection arrangement of a finite Coxeter group $W$.

$$1 \to \mathbf{P}(W) \to \mathbf{B}(W) \to W \to 1$$

$\mathbf{P}(W) := \pi_1(\mathbb{C}^n \setminus \mathcal{H}_{\mathbb{C}}, x_B)$ is the pure braid group of $W$

$\mathbf{B}(W) := \pi_1((\mathbb{C}^n \setminus \mathcal{H}_{\mathbb{C}})/W, x_B)$ is the braid group of $W$

Write $\mathbf{w}$ for the usual lift of $w \in W$ to $\mathbf{B}^+(W)$.

$$\mathsf{Snap}(w) := \mathsf{Pop}(\mathbf{w}) \cdot (\mathbf{w}_\circ(\mathrm{des}(w)))^2.$$

> The map Snap is a poset embedding from $\mathrm{Shard}(W)$ into $[1, \Delta^2]_{\mathbf{B}^+}$.

Interpret everything in $\mathbf{B}(W)$ (since $\mathbf{P}^+(W) \subseteq \mathbf{B}(W)$):

$$\mathsf{Pop}(\mathbf{w}) = \mathbf{w} \cdot \mathbf{w}_\circ(\mathrm{des}(w))^{-1}$$

$$\mathsf{Crackle}(w) = \mathsf{Pop}(\mathbf{w}) \cdot (\mathbf{w}_\circ(\mathrm{des}(w)))^2 \cdot \mathsf{Pop}(\mathbf{w})^{-1}$$

> $\mathsf{Snap}(w) = \mathsf{Crackle}(w) \cdot \mathsf{Pop}(\mathbf{w})$.