Invariant theory for the free left-regular band and a q-analogue

Sarah Brauner [1] **Patricia Commins** [2] Victor Reiner [3]

[1, 2, 3]University of Minnesota, Twin Cities, [1]Max Planck Institute for Mathematics in the Sciences

What are left-regular bands?

Definition. A *left-regular band (LRB)* is a semigroup S satisfying (i) $x^2 = x$ and (ii) $xyx = xy$ for all $x, y \in S$.

Example. The *free LRB on n-letters*, $\mathcal{F}_n$, is the set of words without repeated letters on the alphabet $\{1, 2, \cdots, n\}$ under the operation

$$(u_1, u_2, \cdots, u_k) \cdot (v_1, v_2, \cdots, v_\ell) = (u_1, u_2, \cdots, u_k, v_1, v_2, \cdots, v_\ell)^\wedge,$$

where $\wedge$ denotes deleting any letter which has previously appeared, when reading left-to-right. For example, in $\mathcal{F}_6$,

$$(1, 4, 2, 5) \cdot (2, 1, 6, 3) = (1, 4, 2, 5, 6, 3).$$

Example. The *q-free LRB on n letters*, $\mathcal{F}_n^{(q)}$, is the set of flags $(V_1, V_2, \cdots, V_k)$ of $\mathbb{F}_q^n$ with $0 \le k \le n$ and $\dim V_i = i$, under the operation

$$(U_1, \cdots, U_k) \cdot (V_1, \cdots, V_\ell) = (U_1, U_2, \cdots, U_k, U_k + V_1, U_k + V_2, \cdots, U_k + V_\ell)^\wedge.$$

Both $\mathcal{F}_n, \mathcal{F}_n^{(q)}$ are *monoids*; the identities are the empty word or flag, $()$.

History. Many popular shuffling operators can be realized in LRBs. The combinatorics of LRBs explains the eigenvalues of these operators (see [3]), which in turn reveals information about their long-term behaviors.

Example. The element

$$x := (1) + (2) + \cdots + (n) \in \mathbf{k}\mathcal{F}_n$$

acts like *random-to-top shuffling* on words of length n in $\mathcal{F}_n$. For example,

$$((1) + (2) + (3) + (4)) \cdot (1, 2, 3, 4) = (1, 2, 3, 4) + (2, 1, 3, 4) + (3, 1, 2, 4) + (4, 1, 2, 3).$$

Left-regular bands under symmetry

Many LRBs come equipped with natural group actions. In our cases,

- The symmetric group $\mathfrak{S}_n$ acts on $\mathcal{F}_n$ by
$$\pi(u_1, u_2, \cdots, u_k) = (\pi(u_1), \pi(u_2), \cdots, \pi(u_k)).$$
- The finite general linear group $GL_n(\mathbb{F}_q)$ acts on $\mathcal{F}_n^{(q)}$ by
$$g(U_1, \cdots, U_k) = (g(U_1), \cdots, g(U_k)).$$

Why LRBs under symmetry? In [1], Bidigare proved the *invariant subalgebra* of the face algebra of a reflection arrangement (an LRB algebra) is anti-isomorphic to *Solomon's descent algebra*.

Our two questions

For both monoids $M = \mathcal{F}_n, \mathcal{F}_n^{(q)}$, we examine the algebra $\mathbf{k}M$, and answer the two *main questions of invariant theory* for the corresponding symmetry groups $G = \mathfrak{S}_n$, $GL_n(\mathbb{F}_q)$ acting on $\mathbf{k}M$:

1. What is the structure of the invariant subalgebra $(\mathbf{k}M)^G$?
2. What is the structure of $\mathbf{k}M$, simultaneously as a $(\mathbf{k}M)^G$-module and a G-representation?

Answer to question # 1

Theorem. *Let $\mathbf{k}$ be a commutative ring with 1. Recall the random-to-top element $x \in \mathbf{k}\mathcal{F}_n$.*

(i) *The unique $\mathbf{k}$-algebra map $\mathbf{k}[X] \longrightarrow \mathbf{k}\mathcal{F}_n$ mapping $X \mapsto x$ induces an algebra isomorphism*
$$(\mathbf{k}\mathcal{F}_n)^{\mathfrak{S}_n} \cong \mathbf{k}[X] / \left(X(X-1)(X-2) \cdots (X-n) \right).$$
Hence $(\mathbf{k}M)^{\mathfrak{S}_n}$ is commutative and generated by x.

(ii) *If furthermore $\mathbf{k}$ is a field where $n!$ is invertible, then x acts semisimply on any finite-dimensional $(\mathbf{k}\mathcal{F}_n)^{\mathfrak{S}_n}$-module, with eigenvalues contained in the list $\{0, 1, 2, \cdots, n\}$.*

The q-analogous theorem. If $q \in \mathbf{k}^\times$, then $\left(\mathbf{k}\mathcal{F}_n^{(q)}\right)^{GL_n(\mathbb{F}_q)}$ is a commutative ring generated by $x^{(q)}$, a natural q-analogue of x. If furthermore $\mathbf{k}$ is a field with $|GL_n(\mathbb{F}_q)| \in \mathbf{k}^\times$, then $x^{(q)}$ acts semisimply on any $\left(\mathbf{k}\mathcal{F}_n^{(q)}\right)^{GL_n(\mathbb{F}_q)}$-module, with eigenvalues in $\{[0]_q, [1]_q, [2]_q, \cdots, [n]_q\}$.

The derangement symmetric function

Definition. The descent set, $\mathrm{Des}(Q)$, of a standard Young tableau Q is

$$\mathrm{Des}(Q) := \{i : i+1 \text{ appears in a row strictly below } i \text{ in } Q\}.$$

Example.

$$Q = \begin{array}{|c|c|c|} \hline 1 & 4 & 6 \\ \hline 2 & 5 & 7 \\ \hline 3 \\ \cline{1-1} 8 \\ \cline{1-1} \end{array} \implies \mathrm{Des}(Q) = \{1, 2, 4, 6, 7\}.$$

Definition. A standard Young tableau Q with n cells is a *desarrangement tableau* if the smallest element of $\{1, 2, \ldots, n\} \setminus \mathrm{Des}(Q)$ is even.

The *derangement symmetric function* $\mathfrak{d}_n$, introduced by Désarménien–Wachs [4], can be defined in terms of Schur functions due to Reiner– Webb [5]:

$$\mathfrak{d}_n := \sum_Q s_{\lambda(Q)},$$

where Q runs through the desarrangement tableaux of size n.

Example. Since the desarrangement tableaux of size 5 are

$$\begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 5 \\ \hline 2 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2} \end{array} \quad \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} \end{array} \quad \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|} \hline 1 & 5 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array},$$

the derangement symmetric function for $n = 5$ is

$$\mathfrak{d}_5 = s_{(4,1)} + 2s_{(3,2)} + 2s_{(3,1,1)} + 2s_{(2,2,1)} + 2s_{(2,1,1,1)}.$$

Terminology: *Derangements* are permutations in $\mathfrak{S}_n$ without fixed points and *desarrangements* are permutations in $\mathfrak{S}_n$ for which the first non-descent is even. These two sets are in bijection, having size the dimension of the $\mathfrak{S}_n$-representation corresponding to $\mathfrak{d}_n$.

Answer to question # 2

Theorem. *Let $\mathbf{k}$ be a field with $n! \in \mathbf{k}^\times$. Then x acts semisimply on $\mathbf{k}\mathcal{F}_n$, and for each $j = 0, 1, 2, \ldots, n$, its $j-$ eigenspace carries a $\mathfrak{S}_n$-representation with Frobenius image*

$$\sum_{\ell=j}^{n} h_{(n-\ell, j)} \cdot \mathfrak{d}_{\ell-j}. \tag{1}$$

The q-analogous theorem. There is a q-Frobenius ring isomorphism between *unipotent* representations of $GL_n(\mathbb{F}_q)$ and symmetric functions. When $\mathbf{k}$ is a field with $|GL_n(\mathbb{F}_q)| \in \mathbf{k}^\times$, $x^{(q)}$ acts semisimply on $\mathbf{k}\mathcal{F}_n^{(q)}$ and the $[j]_q$–eigenspace of $x^{(q)}$ has q-Frobenius characteristic equal to (1).

Example. We compute the Frobenius image for each j-eigenspace of x on $\mathbf{k}\mathcal{F}_3$, and also the q-Frobenius image of each $[j]_q$-eigenspace of $x^{(q)}$ on $\mathbf{k}\mathcal{F}_3^{(q)}$.

	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 3$
$j = 0$	$h_3 \cdot \mathfrak{d}_0 +$	$h_2 \cdot \mathfrak{d}_1$	$+ \quad h_1 \cdot \mathfrak{d}_2$	$+ \ h_0 \cdot \mathfrak{d}_3$
$j = 1$		$h_{(2,1)} \cdot \mathfrak{d}_0$	$+ \ h_{(1,1)} \cdot \mathfrak{d}_1$	$+ \ h_1 \cdot \mathfrak{d}_2$
$j = 2$			$h_{(2,1)} \cdot \mathfrak{d}_0$	$+ \ h_2 \cdot \mathfrak{d}_1$
$j = 3$				$h_3 \cdot \mathfrak{d}_0$

Ongoing Related Work

- **General left-regular bands under symmetry** (Commins, thesis work)
- **Eigenvalues of a q-analogue of random-to-random shuffling** (Brauner and Commins, with UMN REU 2022 students Ilani Axelrod-Freed, Judy Chiang, and Veronica Lang)

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Contact Information: **commi010@umn.edu**