Promotion and Growth Diagrams for Fans of Dyck Paths

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Motivating Question

Find a diagrammatic basis for the space of invariant tensors $\mathsf{Inv}(V^{\otimes n})$ that respects the natural cyclic action of C_n and where V is:

- the spin representation of type B
- the vector representation of type C
- the vector representation of type B (not shown on poster)

Highest Weight Words of Weight Zero

Cyclic action on invariant tensors of $V^{\otimes n} \longleftrightarrow$ promotion on HWW_0 in crystal $\mathcal{B}^{\otimes n}$ [12].

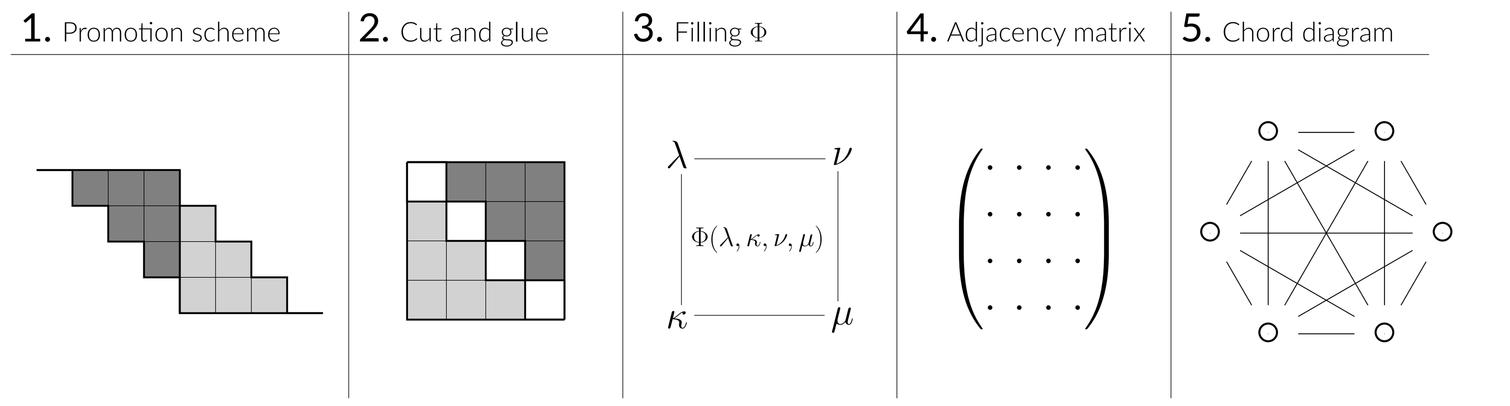
- r-fans of Dyck paths of length $n \longleftrightarrow \mathsf{HWW}_0$ in $\mathcal{B}_{\mathsf{spin}}^{\otimes n}$ of type B_r

$$\mathcal{F} = ((000),(111),(220),(111),(000)) \leftrightarrow (-,-,-)\otimes(-,-,+)\otimes(+,+,-)\otimes(+,+,+) \in \mathcal{B}_{\mathsf{spin}}^{\otimes 4}$$

- r-symplectic oscillating tableaux of length n and weight $\emptyset \longleftrightarrow \mathsf{HWW}_0$ in $\mathcal{C}_{\mathsf{vec}}^{\otimes n}$ of type C_r

$$\mathcal{O} = ((000),(100),(110),(210),(211),(111),(110),(100),(000)) \leftrightarrow \bar{1}\otimes\bar{2}\otimes\bar{3}\otimes\bar{1}\otimes 3\otimes 1\otimes 2\otimes 1 \in \mathcal{C}_{\mathsf{vec}}^{\otimes 8}$$

Chord Diagrams via Promotion



Promotion (via local rules) and Filling Rule

The local rules of Lenart [7] can be stated as follows: four weight vectors $\lambda,\mu,\kappa,\nu \in \Lambda$ depicted in a square diagram $\begin{smallmatrix}\lambda & \text{—} & \nu \\ | & & | \\ \kappa & \text{—} & \mu\end{smallmatrix}$ satisfy the local rule, if $\mu = \mathrm{dom}_W(\kappa+\nu-\lambda)$. Promotion on highest weight elements in minuscule crystals can be defined via the local rules as shown below.

$$\mathsf{pr}(000,111,222,311,220,111,222,111,000) \mapsto (000,111,200,111,200,311,200,111,000)$$

$$\begin{array}{ccccccccccccccc}
111 & \rightarrow & 222 & \rightarrow & 311 & \rightarrow & 220 & \rightarrow & 111 & \rightarrow & 222 & \rightarrow & 111 & \rightarrow & 000 \\
\uparrow & & \uparrow & & \uparrow & 1 & \uparrow & & \uparrow & & \uparrow & & \uparrow & 2 & \uparrow \\
000 & \rightarrow & 111 & \rightarrow & 200 & \rightarrow & 111 & \rightarrow & 200 & \rightarrow & 311 & \rightarrow & 200 & \rightarrow & 111
\end{array}$$

The filling rule for oscillating tableaux and r-fans of Dyck paths is

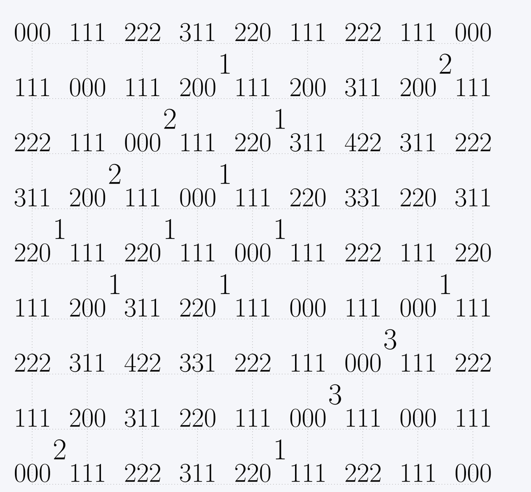
$$\Phi(\lambda,\kappa,\nu,\mu) = \text{ number of negative entries in } \kappa+\nu-\lambda$$

$\mathcal{F} = (000,111,222,311,220,111,222,111,000)$ to Chord Diagram

1) Promotion scheme

```
000 111 222 311 220 111 222 111 000
    000 111 200 111 200 311 200 111 000
        000 111 220 311 422 311 222 111 000
            000 111 220 331 220 311 200 111 000
                000 111 222 111 220 111 220 111 000
                    000 111 000 111 200 311 220 111 000
                        000 111 222 311 422 331 222 111 000
                            000 111 200 311 220 111 000 111 000
                                000 111 222 311 220 111 222 111 000
```

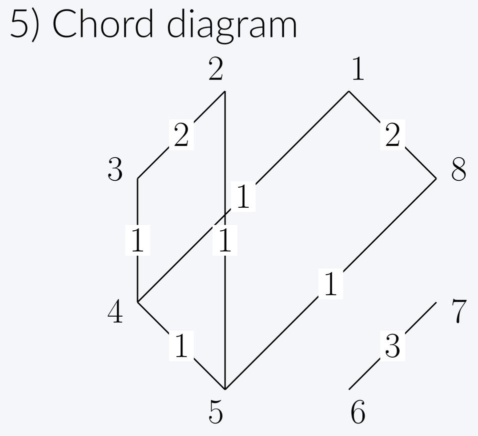
2-3) Cut and glue & Filling



4) Adjacency matrix $\mathsf{M}_F(\mathcal{F})$

$$\begin{pmatrix}
0&0&0&1&0&0&0&2\\
0&0&2&0&1&0&0&0\\
0&2&0&1&0&0&0&0\\
1&0&1&0&1&0&0&0\\
0&1&0&1&0&0&0&1\\
0&0&0&0&0&0&3&0\\
0&0&0&0&0&3&0&0\\
2&0&0&0&1&0&0&0
\end{pmatrix}$$

5) Chord diagram



Virtual crystal $B_r \hookrightarrow C_r$ spin to vector

Define an ordering $<$ on the set $[r]\cup[\bar{r}]$ as follows: $1<2<\cdots<r<\bar{r}<\cdots<\bar{1}$

$$\mathcal{V} := \{v_r\otimes v_{r-1}\otimes\cdots\otimes v_1 \in \widehat{\mathcal{V}} \mid v_i > v_j \text{ and } |v_i| \neq |v_j| \text{ for all } i>j\} \subseteq \mathcal{C}_{\mathsf{vec}}^{\otimes r}$$

Let $f_i = \widehat{f}_i^2$, $e_i = \widehat{e}_i^2$ for $1 \leqslant i < r$ and $f_r = \widehat{f}_r$, $e_r = \widehat{e}_r$.

Proposition: $\mathcal{V}$ is a virtual crystal for the embedding of Lie algebras $B_r \hookrightarrow C_r$ and is isomorphic to $\mathcal{B}_{\mathsf{spin}}$.

This gives a map ι from r-fans of Dyck paths to r symplectic oscillating tableaux.

$$\iota(\mathcal{F}) = ((000),(100),(110),(111),(211),(221),(220),(221),(211),(111),(110),(100),(000))$$

Proposition: For an r-fan of Dyck paths F,

$$\iota\circ\mathsf{pr}(\mathcal{F}) = \mathsf{pr}^r\circ\iota(\mathcal{F})$$

and

$$\mathsf{M}_F(\mathcal{F}) \underset{\mathsf{blocksum}_r}{\overset{\mathsf{blowup}_r^{\mathsf{SE}}}{\rightleftarrows}} \mathsf{M}_O(\iota(\mathcal{F})).$$

Chord Diagrams via Fomin Growth Diagrams

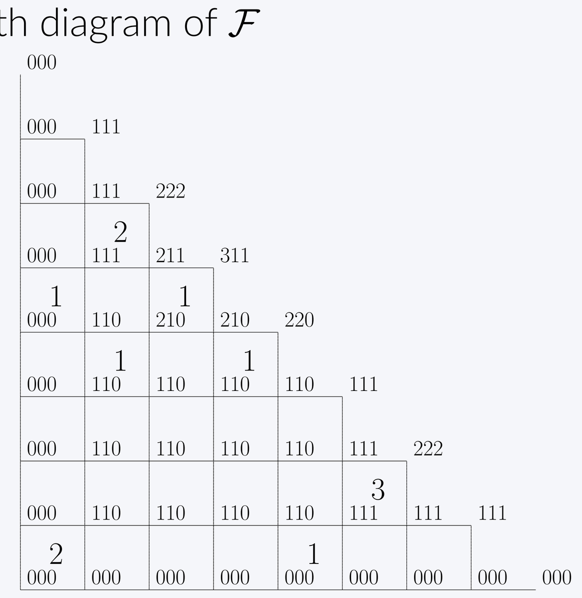
Given a square diagram with filling $\begin{smallmatrix}\alpha & \text{—} & \beta \\ | & \bullet & | \\ \gamma & \text{—} & \delta\end{smallmatrix}$, the local growth rules for oscillating tableaux are given by: (F1) $\beta' = \mathrm{sort}(\alpha'+\gamma'-\lambda')$ if the filling is 0 (F2) $\beta = \gamma + e_1$ if the filling is 1.

To construct the adjacency matrix $\mathsf{G}_O(\mathcal{O})$ or $\mathsf{G}_F(\mathcal{F})$:

1) Label the hypotenuse of a staircase Ferrer shape with $\mathcal{O}$ or $\mathcal{F}$ respectively.
2) Apply the inverse local growth rules.
3) Turn the growth diagram filling into a symmetric matrix.

Growth Diagram of $\mathcal{F} = (000,111,222,311,220,111,222,111,000)$

1-2) Growth diagram of $\mathcal{F}$



3) Adjacency matrix $\mathsf{G}_F(\mathcal{F})$

$$\begin{pmatrix}
0&0&0&1&0&0&0&2\\
0&0&2&0&1&0&0&0\\
0&2&0&1&0&0&0&0\\
1&0&1&0&1&0&0&0\\
0&1&0&1&0&0&0&1\\
0&0&0&0&0&0&3&0\\
0&0&0&0&0&3&0&0\\
2&0&0&0&1&0&0&0
\end{pmatrix}$$

Main Result

Theorem: For an oscillating tableau of weight zero $\mathcal{O}$ and an r-fan of Dyck paths $\mathcal{F}$, we have

$$\mathsf{M}_O(\mathcal{O}) = \mathsf{G}_O(\mathcal{O}) \quad\text{and}\quad \mathsf{M}_F(\mathcal{F}) = \mathsf{G}_F(\mathcal{F}).$$

Corollary: The maps M_O and M_F are injective and intertwine promotion and rotation.

Cyclic Sieving Phenomenon

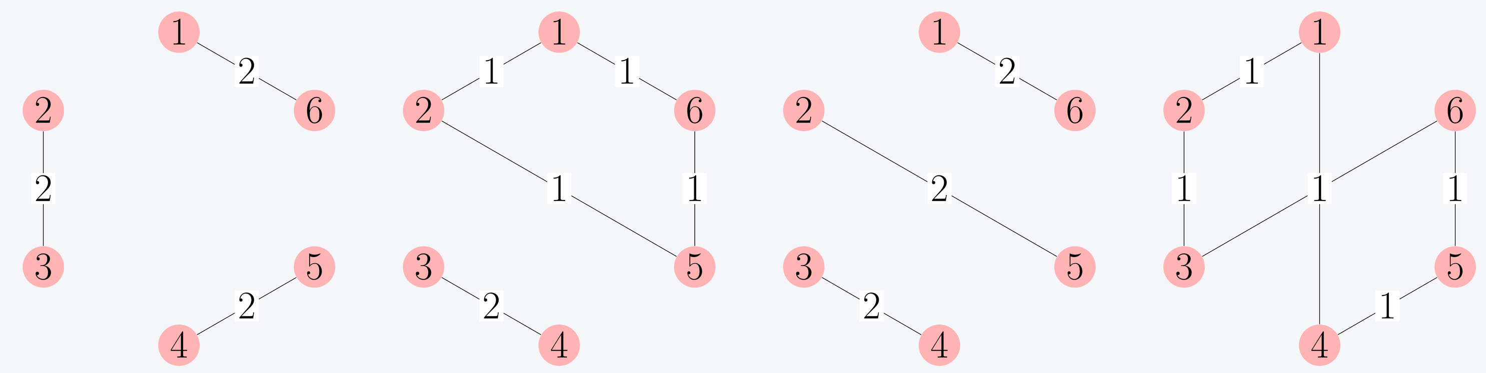
Let X be a set with an action of the cyclic group $C_n = \langle g\rangle$ and $f(q)\in\mathbb{N}[q]$. The triple $(X, C_n, f(q))$ exhibits the cyclic sieving phenomenon (CSP) if $f(e^{\frac{2d\pi i}{n}}) = |\{x\in X | g^d\cdot x = x\}|$ for all $d\in\mathbb{N}$.

Theorem [3, 12]: Let X be the set of HWW_0 in $\mathcal{B}^{\otimes n}$ where $\mathcal{B}$ is minuscule. Then the triple $(X, C_n, f_{n,r}(q))$ exhibits the CSP where C_n acts by promotion and $f_{n,r}$ is the energy generating function over the set of HWW_0 in $\mathcal{B}^{\otimes n}$.

Conjecture: The triple $(\mathcal{F}_{2n,r}, C_{2n}, g_{n,r})$ exhibits the CSP where $\mathcal{F}_{2n,r}$ is the set of r-fans of Dyck paths of length $2n$, C_{2n} acts via promotion, and $g_{n,r} = \prod_{1\leqslant i\leqslant j\leqslant n-1}\frac{[i+j+2r]_q}{[i+j]_q}$.

CSP Example for $\mathcal{F}_{6,2}$

Orbit representatives after applying M_F:



Generating functions:

$$q^{-6}f_{6,2}(q) = q^{10}+q^9+2q^8+q^7+3q^6+q^5+2q^4+q^3+q^2+1$$
$$g_{3,2}(q) = q^{12}+q^{10}+q^9+2q^8+q^7+2q^6+q^5+2q^4+q^3+q^2+1$$
$$f_{6,2}(q) \equiv g_{3,2}(q) \equiv q^5+3q^4+2q^3+3q^2+q+4 \ \mathsf{mod}(q^6-1)$$

References

[1] Daniel Bump and Anne Schilling. *Crystal bases*. World Scientific Publ. Co. Pte. Ltd., Hackensack, NJ, 2017. Representations and combinatorics.

[2] S. V. Fomin. The generalized Robinson-Schensted-Knuth correspondence. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 155(Differentsial' naya Geometriya, Gruppy Li i Mekh. VIII):156–175, 195, 1986.

[3] Bruce Fontaine and Joel Kamnitzer. Cyclic sieving, rotation, and geometric representation theory. *Selecta Math. (N.S.)*, 20(2):609–625, 2014.

[4] André Henriques and Joel Kamnitzer. Crystals and coboundary categories. *Duke Math. J.*, 132(2):191–216, 2006.

[5] Sam Hopkins. Order polynomial product formulas and poset dynamics. preprint `arXiv:2006.01568`, 2020.

[6] C. Krattenthaler. Growth diagrams, and increasing and decreasing chains in fillings of Ferrers shapes. *Adv. in Appl. Math.*, 37(3):404–431, 2006.

[7] Cristian Lenart. On the combinatorics of crystal graphs. II. The crystal commutor. *Proc. Amer. Math. Soc.*, 136(3):825–837, 2008.

[8] Stephan Pfannerer, Martin Rubey, and Bruce Westbury. Promotion on oscillating and alternating tableaux and rotation of matchings and permutations. *Algebr. Comb.*, 3(1):107–141, 2020.

[9] Thomas Walton Roby, V. *Applications and extensions of Fomin's generalization of the Robinson-Schensted correspondence to differential posets*. ProQuest LLC, Ann Arbor, MI, 1991. Thesis (Ph.D.)–Massachusetts Institute of Technology.

[10] Sheila Sundaram. Orthogonal tableaux and an insertion algorithm for $\mathrm{SO}(2n+1)$. *J. Combin. Theory Ser. A*, 53(2):239–256, 1990.

[11] Marc A. A. van Leeuwen. An analogue of jeu de taquin for Littelmann's crystal paths. *Sém. Lothar. Combin.*, 41:Art. B41b, 23 pp., 1998.

[12] Bruce W. Westbury. Invariant tensors and the cyclic sieving phenomenon. *Electron. J. Combin.*, 23(4):Paper 4.25, 40, 2016.

[13] Bruce W. Westbury. Coboundary categories and local rules. *Electron. J. Combin.*, 25(4):Paper No. 4.9, 22, 2018.

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