A Realization of Poset Associahedra

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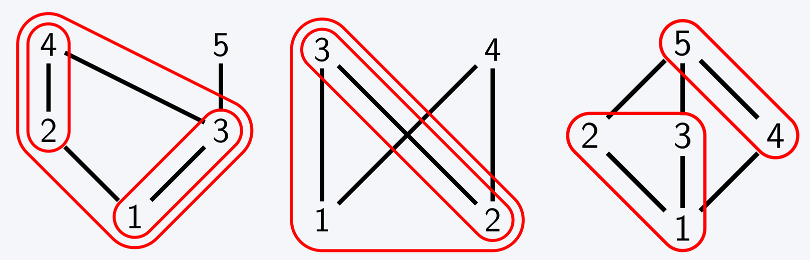
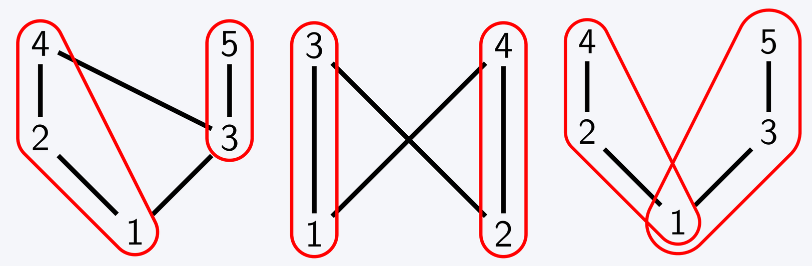
Background: Tubes and Tubings

- A *proper tube* of a poset P is a connected, convex subset $\tau \subset P$ such that $1 < |\tau| < |P|$.
- For disjoint tubes σ, τ we say $\tau \prec \sigma$ if there exists $a \in \tau, b \in \sigma$ such that $a \prec b$.
- A *proper tubing* T of P is a set of proper tubes of P such that any pair of tubes is nested or disjoint and that the relation $\prec$ is acyclic.

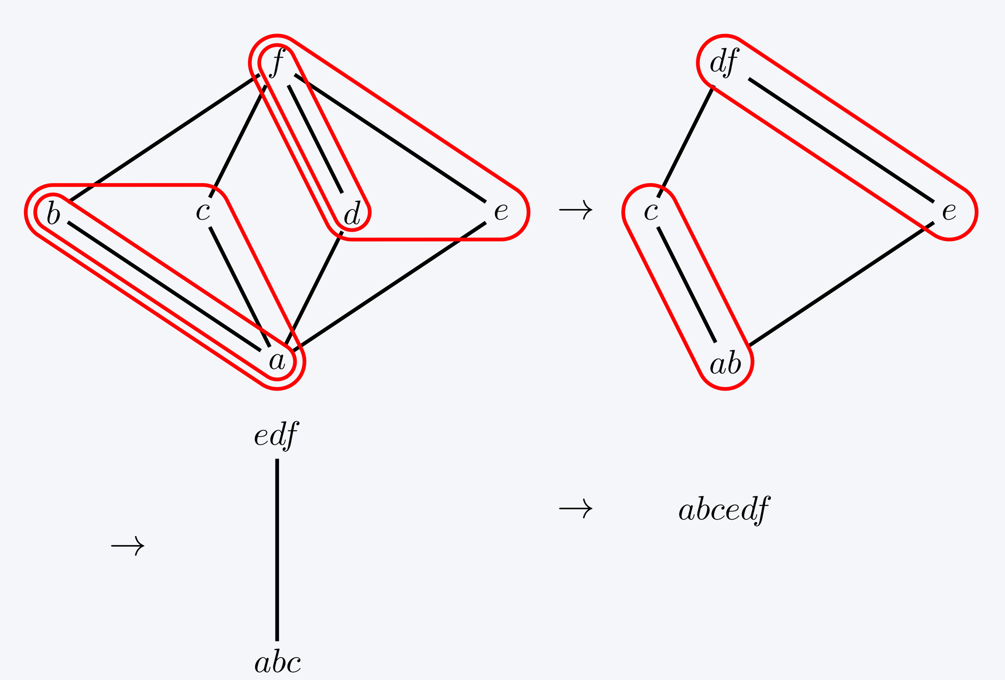


Examples



Non-examples

Background: Maximal tubings as bracketings



An evaluation of a poset along a tubing

Background: Poset Associahedra

Theorem (Galashin (2021)) For a finite, connected poset P, the *poset associahedron* $\mathscr{A}(P)$ is a simple, convex polytope of dimension $|P|-2$ whose face lattice is isomorphic to the set of proper tubings ordered by reverse inclusion.

Problem (Galashin (2021)) Provide an explicit realization of poset associahedra.

Shapiro's Catalan Convolution

Let $B_n = \binom{2n}{n}$ and $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Theorem (Shapiro (2002), Andrews (2011), Nagy (2012))

$$\sum_{i=0}^{2n} C_{2i}C_{2(n-i)} = 4^n C_n \text{ and } \sum_{i=0}^{2n} B_{2i}C_{2(n-i)} = 4^n B_n.$$

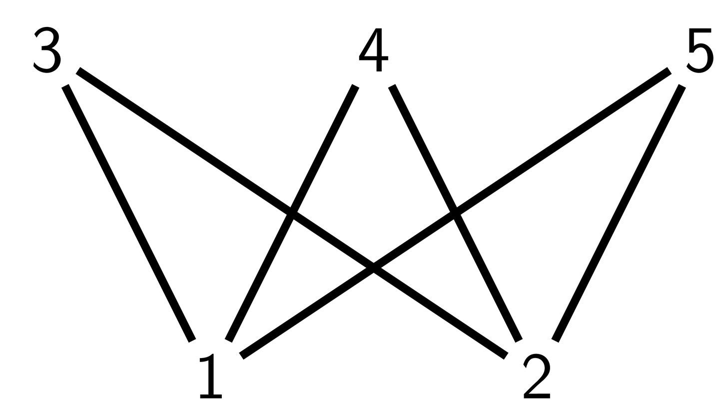
This is trivial to prove with generating functions, but complicated to prove bijectively.

Main Result: A Realization

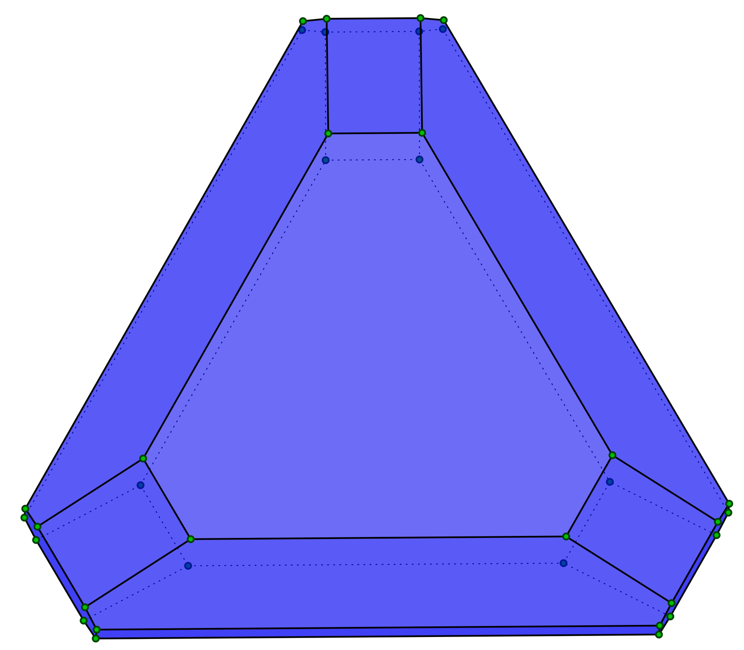
Definition Let P be a finite, connected poset and let $\tau \subseteq P$. We define:

- The *order cone*
$$\mathscr{L}(P) := \left\{x \in \mathbb{R}^P_{\Sigma=0} \mid x_i \le x_j \text{ for all } i \preceq_P j\right\}$$
- $\alpha_\tau : \mathscr{L}(P) \to \mathbb{R}$ by
$$\alpha_\tau(x) := \sum_{\substack{i \prec j \\ i,j \in \tau}} x_j - x_i$$
- The half-space
$$h_\tau := \left\{x \in \mathbb{R}^P_{\Sigma=0} \mid \alpha_\tau(x) \ge n^{2|\tau|}\right\}$$
- The hyperplane
$$H_\tau := \left\{x \in \mathbb{R}^P_{\Sigma=0} \mid \alpha_\tau(x) = n^{2|\tau|}\right\}$$

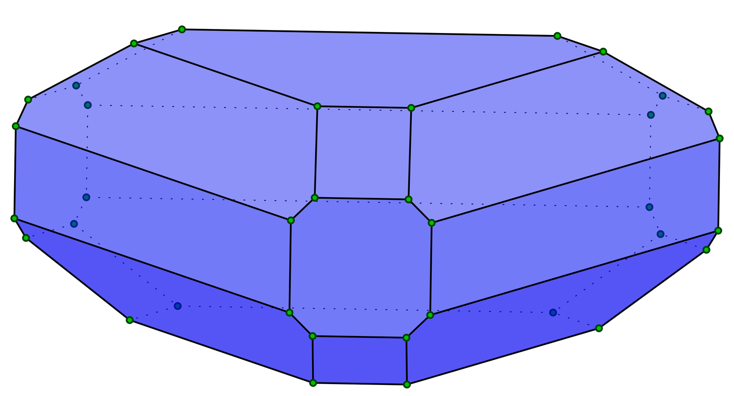
Theorem (S. (2022)) If P is a finite, connected poset, then the intersection of H_P with h_τ for all proper tubes τ gives a realization of $\mathscr{A}(P)$.



$K_{2,3}$

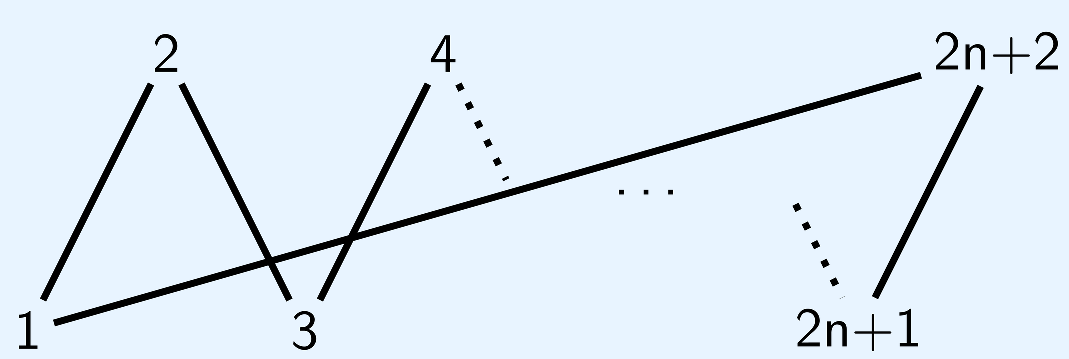


Top view of $\mathscr{A}(K_{2,3})$



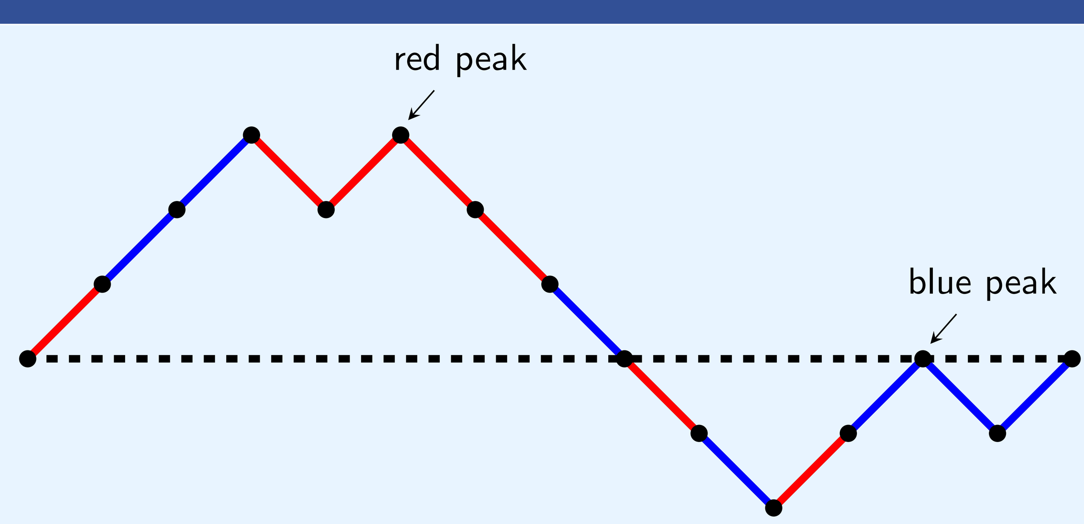
Side view of $\mathscr{A}(K_{2,3})$

A Mysterious Example



The cyclic fence poset $CF_{2(n+1)}$

$CFA_{2n} := \mathscr{A}\left(CF_{2(n+1)}\right)$ has $\sum_{i=0}^{2n} B_{2i}C_{2(n-i)}$ vertices.



A colored balanced path (CBP)

There are $4^n\binom{2n}{n}$ CBPs of length $2n$.

Catalan Convolution: A Refinement

The vertices of CFA_{2n} admit a refinement: orient the edges according to a generic direction. It is known that

$$h_{CFA_{2n}}(t) = \sum_{v \in \text{Vertices}} t^{\text{outdegree}(v)}$$

CBPs also admit a refinement. For a CBP P of length $2n$, let

$$\text{stat}(P) = n + \#\text{red peaks} - \#\text{blue peaks}$$

Proposition (S. (2022))

$$h_{CFA_{2n}}(t) = \sum_{P \in CBP_{2n}} t^{\text{stat}(P)}$$

CFA: A type A analogue?

The *cyclohedron* $\mathscr{C}_n$ is a cyclic version of the associahedron $\mathscr{A}_n$. This raises the question: What if we color Dyck paths instead?

Defining stat(P) similarly, we get:

	f_0	h_i	γ_i	Path Type
$\mathscr{A}_{n+1}$	C_n	$N(n,i)$		Dyck
???	$4^n C_n$		$4^i N(n,i)$	Colored Dyck
$\mathscr{C}_{n+1}$	B_n	$\binom{n}{i}^2$		Balanced
CFA_{2n}	$4^n B_n$		$4^i \binom{n}{i}^2$	Colored Balanced

where $N(n,i) = \frac{1}{n}\binom{n}{i}\binom{n}{i-1}$ are the *Narayana numbers*.

Proof Idea: Key Lemma

Definition For $S \subseteq P$ and $x \in \mathbb{R}^P$, define the *diameter* of x relative to S by

$$\text{diam}_S(x) = \max_{i,j \in S} |x_i - x_j|.$$

Equivalently, $\text{diam}_S(x)$ is the diameter of $\{x_i : i \in S\}$ as a subset of $\mathbb{R}$.

Lemma Let $\tau \subseteq P$ be a tube and let $x \in \mathscr{L}(P)$. Then

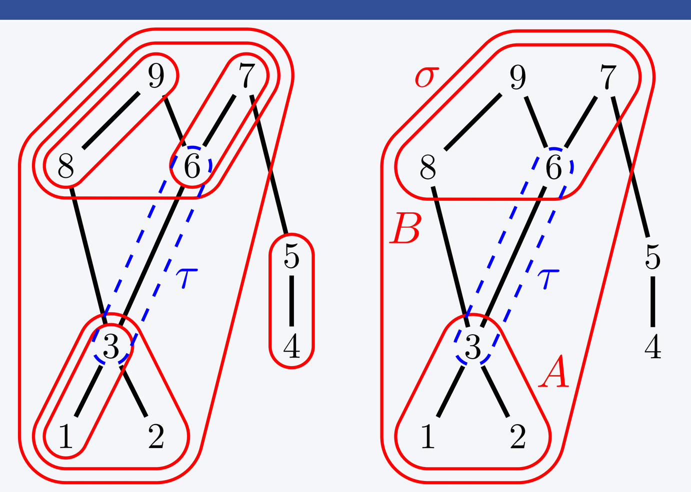
$$\text{diam}_\tau(x) \le \alpha_\tau(x) \le \frac{n^2}{4}\text{diam}_\tau(x).$$

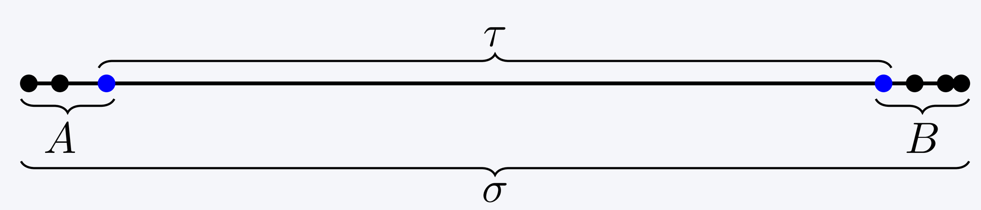
That is, α_τ is an approximation of diam_τ.

Main Idea of Realization

Let $\sigma, \tau \subseteq P$ be tubes with $|\tau| < |\sigma|$. Then for $x \in H_\tau$ and $y \in H_\sigma$, $\text{diam}_\tau(x)$ is exponentially smaller than $\text{diam}_\sigma(y)$.

Proof Illustration





Let T be a maximal tubing and $x = \bigcap_{\tau \in T} H_\tau$. Then if $\sigma \notin T$ is a tube, $\text{diam}_\sigma(x)$ must be large. Hence x lies in the interior of h_σ.

Open Questions

1. Given a combinatorial interpretation of the h-vector of poset associahedra.
2. Are the h-polynomials of poset associahedron all real-rooted?
3. Can the refinement of the Catalan convolution be proven bijectively?
4. Is there a natural polytope that fills in missing box of the table?

References

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