BARRETT-JOHNSON INEQUALITIES FOR TOTALLY NONNEGATIVE MATRICES

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Outline
(1) HPSD and TNN matrices
(2) Classical inequalities
(3) The Barrett-Johnson inequality for real PSD matrices
(4) Extension to TNN matrices
(5) Open problems
HPSD and TNN matrices

Given $n \times n$ matrix $A = (a_{i,j})$, subsets $I, J \subset [n] := \{1, \ldots, n\}$, define submatrix $A_{I,J} = (a_{i,j})_{i \in I, j \in J}$.

Call $A$ Hermitian (H) if $A^* = A$,
positive semidefinite (PSD) if $x^*Ax \geq 0$ for all $x \in \mathbb{C}^n$,
where * denotes conjugate transpose.

Call $A$ totally nonnegative (TNN) if
$\det(A_{I,J}) \geq 0$ for all $I, J \subseteq [n], \ |I| = |J|$.

Fact: $A$ is HPSD if $a_{j,i}^* = a_{i,j}$ for all $i, j \in [n],
\det(A_{I,I}) \geq 0$ for all $I \subseteq [n]$. 
Classical inequalities for $A$ HPSD or TNN

Hadamard (HPSD), Koteljanskii (TNN):
$$\det(A) \leq a_{1,1} \cdots a_{n,n}.$$ 

Fischer (HPSD), Fan (TNN): For all $I \subseteq [n]$, $\overline{I} := [n] \setminus I$,
$$\det(A) \leq \det(A_{I,I}) \det(A_{\overline{I},\overline{I}}).$$

Schur (HPSD), Stembridge (TNN): For all $\mathfrak{S}_n$-characters $\chi$,
$$\det(A) \leq \frac{\operatorname{Imm}_\chi(A)}{\chi(e)},$$
where
$$\operatorname{Imm}_\chi(A) := \sum_{w \in \mathfrak{S}_n} \chi(w)a_{1,w(1)} \cdots a_{1,w(n)}.$$
Barrett-Johnson inequalities for $A$ real PSD

Given $\lambda = (\lambda_1, \ldots, \lambda_r), \mu = (\mu_1, \ldots, \mu_s) \vdash n$, we have

$$\sum \frac{\det(A_{I_1,I_1}) \cdots \det(A_{I_r,I_r})}{\binom{n}{\lambda_1, \ldots, \lambda_r}} \leq \sum \frac{\det(A_{J_1,J_1}) \cdots \det(A_{J_s,J_s})}{\binom{n}{\mu_1, \ldots, \mu_s}},$$

for all $A$ real PSD if and only if $\lambda \succeq \mu$ in dominance.

Sums are over ordered set partitions of $[n]$ of type $\lambda$ and $\mu$; the numbers of these are

$$\begin{align*}
\binom{n}{\lambda_1, \ldots, \lambda_r} &= \frac{n!}{\lambda_1! \cdots \lambda_r!}, \\
\binom{n}{\mu_1, \ldots, \mu_s} &= \frac{n!}{\mu_1! \cdots \mu_s!}.
\end{align*}$$
Main result

**Theorem:** (SS ’22) The Barrett-Johnson inequalities hold for TNN matrices.

**Example:** For $n = 4$ we have $4 \succeq 31 \succeq 22 \succeq 211 \succeq 1111$ and

\[
\begin{align*}
\det(A) & \leq \frac{\det(A_{123,123})a_{4,4} + \cdots + \det(A_{234,234})a_{1,1}}{4} \\
& \leq \frac{\det(A_{12,12})\det(A_{34,34}) + \cdots + \det(A_{34,34})\det(A_{12,12})}{6} \\
& \leq \frac{\det(A_{12,12})a_{3,3}a_{4,4} + \cdots + \det(A_{34,34})a_{2,2}a_{1,1}}{12} \\
& \leq \frac{a_{1,1}a_{2,2}a_{3,3}a_{4,4} + \cdots + a_{4,4}a_{3,3}a_{2,2}a_{1,1}}{24}.
\end{align*}
\]
\[
\begin{array}{c}
\left( \frac{1}{4} \right) \\
\left( \frac{1}{6} \right) \\
\left( \frac{1}{12} \right) \\
\left( \frac{1}{24} \right)
\end{array}
\]
Connection to induced sign characters

Define the *induced trivial* and *induced sign* characters by
\[ \eta^\lambda = \text{triv} \uparrow_{S^n}^{S_\lambda}, \quad \epsilon^\lambda = \text{sgn} \uparrow_{S^n}^{S_\lambda}. \]

**Theorem:** (Littlewood ’40, Merris–Watkins ’85)

\[ \text{Imm}_{\epsilon^\lambda}(A) = \sum_{(I_1,\ldots,I_r)} \det(A_{I_1,I_1}) \cdots \det(A_{I_r,I_r}), \]
\[ \text{Imm}_{\eta^\lambda}(A) = \sum_{(I_1,\ldots,I_r)} \text{per}(A_{I_1,I_1}) \cdots \text{per}(A_{I_r,I_r}), \]

summed over ordered set partitions of type \( \lambda = (\lambda_1, \ldots, \lambda_r) \).

Barrett-Johnson inequalities can be rewritten as
\[ \frac{1}{n^{\lambda_1,\ldots,\lambda_r}} \text{Imm}_{\epsilon^\lambda}(A) \leq \frac{1}{n^{\mu_1,\ldots,\mu_s}} \text{Imm}_{\epsilon^\mu}(A). \]
Proof idea

It suffices to show that for all TNN $A$ and $k \leq \lfloor \frac{n}{2} \rfloor - 1$ we have

$$\sum_{|I|=k} \frac{\det(A_{I,I}) \det(A_{\bar{I},\bar{I}})}{\binom{n}{k}} \leq \sum_{|I|=k+1} \frac{\det(A_{I,I}) \det(A_{\bar{I},\bar{I}})}{\binom{n}{k+1}},$$

equivalently,

$$\frac{\operatorname{Imm}_{\epsilon n-k-1,k+1}(A)}{\binom{n}{k+1}} - \frac{\operatorname{Imm}_{\epsilon n-k,k}(A)}{\binom{n}{k}} \geq 0. \quad (1)$$

We prove (1) in two ways:

1. using monomial trace immanants,
2. using Temperley-Lieb immanants.
Monomial trace version of proof

Let \( \{ \phi^\lambda \mid \lambda \vdash n \} \) be the monomial traces of \( \mathfrak{S}_n \),

\[
s_\lambda = \sum_{\mu \vdash n} K_{\lambda,\mu} m_\mu, \quad \chi^\lambda = \sum_{\mu \vdash n} K_{\lambda,\mu} \phi^\mu.
\]

**Conjecture:** (Stembridge, '91) For all \( n \times n \) TNN matrices \( A \) and all partitions \( \lambda \vdash n \) we have \( \text{Imm}_{\phi^\lambda}(A) \geq 0 \).

**Theorem:** (CHSS, '16) For all \( n \times n \) TNN matrices \( A \) and all partitions \( \lambda \vdash n \) with \( \lambda_1 \leq 2 \) we have \( \text{Imm}_{\phi^\lambda}(A) \geq 0 \).

**Proposition:** There are \( \{ c_{k,\mu} \mid \mu \vdash n, \mu_1 \leq 2 \} \subseteq \mathbb{N} \) satisfying

\[
\frac{\text{Imm}_{\epsilon^{n-k-1,k+1}}(A)}{\binom{n}{k+1}} - \frac{\text{Imm}_{\epsilon^{n-k,k}}(A)}{\binom{n}{k}} = \sum_{\substack{\mu \vdash n \\ \mu_1 \leq 2}} c_{k,\mu} \text{Imm}_{\phi^\mu}(A).
\]
Temperley-Lieb version of proof

The Temperley-Lieb algebra $T_n(2)$ is generated over $\mathbb{C}$ by $t_1, \ldots, t_{n-1}$ subject to relations

$$
t_i^2 = 2t_i \quad i = 1, \ldots, n - 1,
$$
$$
t_i t_j t_i = t_i \quad \text{if } |i - j| = 1,
$$
$$
t_i t_j = t_j t_i \quad \text{if } |i - j| \geq 2.
$$

Using $\mathfrak{S}_n$-generators $s_1, \ldots, s_{n-1}$ define the map

$$
\sigma : \mathbb{C}[\mathfrak{S}_n] \rightarrow T_n(2),
$$

$$
{s}_i \mapsto t_i - 1.
$$

This is surjective with kernel equal to the ideal

$$(1 + s_1 + s_2 + s_1 s_2 + s_2 s_1 + s_1 s_2 s_1),$$

and image spanned by the standard basis of $T_n(2)$

$$
\mathcal{B}_n = \{ \sigma(w) \mid w \in \mathfrak{S}_n \text{ avoids the pattern } 321 \}.
$$
For each $\tau \in \mathcal{B}_n$, define a linear functional
\[
f_\tau : \mathbb{C}[\mathfrak{S}_n] \to \mathbb{R},
\]
\[
w \mapsto \text{coefficient of } \tau \text{ in } \sigma(w).
\]
and the immanant $\text{Imm}_\tau(A) := \text{Imm}_{f_\tau}(A)$.

**Theorem:** (RS, ’05) For all $n \times n$ TNN matrices $A$ and all $\tau \in \mathcal{B}_n$ we have $\text{Imm}_\tau(A) \geq 0$.

**Proposition:** There are $\{d_{k,\tau} \mid \tau \in \mathcal{B}_n\} \subseteq \mathbb{N}$ satisfying
\[
\frac{\text{Imm}_{\epsilon n-k-1,k+1}(A)}{\binom{n}{k+1}} - \frac{\text{Imm}_{\epsilon n-k,k}(A)}{\binom{n}{k}} = \sum_{\tau \in \mathcal{B}_n} d_{k,\tau} \text{Imm}_\tau(A).
\]
More classical inequalities for $A$ HPSD or TNN

Marcus (HPSD), clear (TNN):

$$\text{per}(A) \geq a_{1,1} \cdots a_{n,n}.$$ 

Lieb (HPSD), clear (TNN): For all $I \subseteq [n],$

$$\text{per}(A) \geq \text{per}(A_{I,I})\text{per}(A_{\overline{I},\overline{I}}).$$

 Conj. by Lieb (HPSD), Johnson (TNN): For all $\mathfrak{S}_n$-characters $\chi,$

$$\text{per}(A) \geq \frac{\text{Imm}_\chi(A)}{\chi(e)}.$$
Open problems

Problem: Show that the Barrett-Johnson inequalities hold for all HPSD matrices.

Problem: Characterize the pairs $(\lambda, \mu)$ of partitions of $n$ for which a permanental analog of the Barrett-Johnson inequalities

$$(2) \quad \frac{1}{n} \text{Imm}_{\eta^\lambda}(A) \geq \frac{1}{n} \text{Imm}_{\eta^\mu}(A).$$

holds for all HPSD or real PSD matrices.

Problem: Characterize the pairs $(\lambda, \mu)$ of partitions of $n$ for which (2) holds for all TNN matrices.