Refined canonical stable Grothendieck polynomials and their duals

Byung-Hak Hwang, Jihyeug Jang*, Jang Soo Kim, Minho Song, and U-Keun Song

Sungkyunkwan University, South Korea

Overview

- We introduce refined canonical stable Grothendieck polynomials and their duals.
- We give Jacobi–Trudi-like formulas, and combinatorial models, Schur expansions, Schur positivity, and dualities of them.

Grothendieck polynomials

- Grothendieck polynomials were introduced for studying the Grothendieck ring of vector bundles on a flag variety.

Grothendieck by Lascoux and Schüzenberger (1982)
+ **stable** by Fomin and Kirillov (1994)
+ **canonical** by Yeliussizov (2017)

$$G_\lambda^{(\alpha,\beta)}(\boldsymbol{x}_n) = \frac{\det\left(x_j^{\lambda_i+n-i}(1+\beta x_j)^{i-1}(1-\alpha x_j)^{-\lambda_i}\right)_{1\le i,j\le n}}{\prod_{1\le i<j\le n}(x_i-x_j)}.$$

Notations

Complete homogeneous symmetric function

$$h_n(\boldsymbol{x}) = \sum_{i_1\le\cdots\le i_n} x_{i_1}\cdots x_{i_n}.$$

Elementary symmetric function

$$e_n(\boldsymbol{x}) = \sum_{i_1<\cdots<i_n} x_{i_1}\cdots x_{i_n}.$$

Plethystic substitutions

For any symmetric function $f(\boldsymbol{x})$,

$$f[z_1+z_2+\cdots] = f(z_1,z_2,\dots).$$

For any formal power series Y and Z,

$$h_n[-Z] = (-1)^n e_n[Z], \qquad e_n[-Z] = (-1)^n h_n[Z],$$

and

$$h_n[Y+Z] = \sum_{a+b=n} h_a[Y]h_b[Z], \qquad e_n[Y+Z] = \sum_{a+b=n} e_a[Y]e_b[Z].$$

New operation $\ominus$

We define

$$h_n[Y\ominus Z] = \sum_{a-b=n} h_a[Y]h_b[Z], \qquad e_n[Y\ominus Z] = \sum_{a-b=n} e_a[Y]e_b[Z].$$

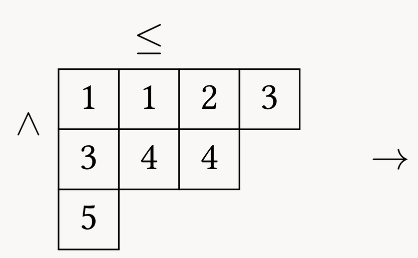
Schur functions

Definition

$$s_\lambda(\boldsymbol{x}_n) = \frac{\det(x_j^{\lambda_i+n-i})_{1\le i,j\le n}}{\prod_{1\le i<j\le n}(x_i-x_j)}.$$

Combinatorial interpretations

$$s_\lambda(\boldsymbol{x}) = \sum_{T\in\mathrm{SSYT}(\lambda)} \boldsymbol{x}^T.$$



$\rightarrow\ \mathrm{wt} = x_1^2x_2x_3^2x_4^2x_5$

Hall inner product

$$\langle s_\lambda(\boldsymbol{x}), s_\mu(\boldsymbol{x})\rangle = \delta_{\lambda,\mu}.$$

Jacobi–Trudi formula / dual Jacobi–Trudi formula

$$s_{\lambda/\mu}(\boldsymbol{x}) = \det\left(h_{\lambda_i-\mu_j-i+j}(\boldsymbol{x})\right)_{i,j=1}^{\ell(\lambda)},$$
$$s_{\lambda'/\mu'}(\boldsymbol{x}) = \det\left(e_{\lambda_i-\mu_j-i+j}(\boldsymbol{x})\right)_{i,j=1}^{\ell(\lambda)}.$$

Involution ω

$$\omega\left(s_{\lambda/\mu}(\boldsymbol{x})\right) = s_{\lambda'/\mu'}(\boldsymbol{x}).$$

Main definitions (Refined canonical stable Grothendieck polynomials)

Let $A_k = \alpha_1+\cdots+\alpha_k$ and $B_k = \beta_1+\cdots+\beta_k$.

$$G_\lambda(\boldsymbol{x}_n;\boldsymbol{\alpha},\boldsymbol{\beta}) := \frac{\det\left(x_j^{\lambda_i+n-i}\prod_{\ell=1}^{i-1}(1-\beta_\ell x_j)\prod_{\ell=1}^{\lambda_i}(1-\alpha_\ell x_j)^{-1}\right)_{1\le i,j\le n}}{\prod_{1\le i<j\le n}(x_i-x_j)}$$
$$= \frac{\det\left(h_{\lambda_i+n-i}[x_j\ominus(A_{\lambda_i}-B_{i-1})]\right)_{1\le i,j\le n}}{\prod_{1\le i<j\le n}(x_i-x_j)},$$

$$g_\lambda(\boldsymbol{x}_n;\boldsymbol{\alpha},\boldsymbol{\beta}) := \frac{\det(h_{\lambda_i+n-i}[x_j-A_{\lambda_i-1}+B_{i-1}])_{1\le i,j\le n}}{\prod_{1\le i<j\le n}(x_i-x_j)}.$$

Remark

Let $\mathbf{0}=(0,0,\dots)$, $\mathbf{1}=(1,1,\dots)$, $\boldsymbol{\alpha}_0=(\alpha,\alpha,\dots)$ and $\boldsymbol{\beta}_0=(\beta,\beta,\dots)$. Our generalizations $G_\lambda(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta})$ and $g_\lambda(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta})$ generalize several well-studied variations of Grothendieck polynomials:

Variations of G and g	introduced by	how to specialize
$G_{\nu/\lambda}(\boldsymbol{x})$	Buch (2002)	$G_{\nu/\lambda}(\boldsymbol{x};\mathbf{0},\mathbf{1})$
$g_{\lambda/\mu}(\boldsymbol{x})$	Lam–Pylyavskyy (2007)	$g_{\lambda/\mu}(\boldsymbol{x};\mathbf{0},\mathbf{1})$
$G_\lambda^{(\alpha,\beta)}(\boldsymbol{x})$	Yeliussizov (2017)	$G_\lambda(\boldsymbol{x};\boldsymbol{\alpha}_0,-\boldsymbol{\beta}_0)$
$g_\lambda^{(\alpha,\beta)}(\boldsymbol{x})$	Yeliussizov (2017)	$g_\lambda(\boldsymbol{x};-\boldsymbol{\alpha}_0,\boldsymbol{\beta}_0)$
$RG_\sigma(\boldsymbol{x};\boldsymbol{\beta})$	Chan–Pflueger (2021)	$G_\sigma(\boldsymbol{x};\mathbf{0},\boldsymbol{\beta})$
$\tilde{g}_{\lambda/\mu}(\boldsymbol{x};\boldsymbol{\beta})$	Galashin–Grinberg–Liu (2016)	$g_{\lambda/\mu}(\boldsymbol{x};\mathbf{0},\boldsymbol{\beta})$
$G_{\lambda/\mu,f/g}(\boldsymbol{x})$	Matsumura (2018)	$G_{\lambda/\mu}^{\mathrm{row}(g,f)}(\boldsymbol{x};\mathbf{0},-\boldsymbol{\beta}_0)$
$\tilde{g}_{\lambda/\mu}^{\mathrm{row}(\boldsymbol{r},\boldsymbol{s})}(\boldsymbol{x};\boldsymbol{\beta})$	Grinberg (private communication), Kim (2022)	$g_{\lambda/\mu}^{\mathrm{row}(\boldsymbol{r},\boldsymbol{s})}(\boldsymbol{x};\mathbf{0},\boldsymbol{\beta})$

Schur expansions

Let $\lambda\in\mathrm{Par}_n$. We have

$$G_\lambda(\boldsymbol{x}_n;\boldsymbol{\alpha},\boldsymbol{\beta}) = \sum_{\mu\supseteq\lambda} C_{\lambda,\mu}(\boldsymbol{\alpha},\boldsymbol{\beta})s_\mu(\boldsymbol{x}_n), \qquad g_\lambda(\boldsymbol{x}_n;\boldsymbol{\alpha},\boldsymbol{\beta}) = \sum_{\mu\subseteq\lambda} c_{\lambda,\mu}(\boldsymbol{\alpha},\boldsymbol{\beta})s_\mu(\boldsymbol{x}_n),$$

where

$$C_{\lambda,\mu}(\boldsymbol{\alpha},\boldsymbol{\beta}) = \det\left(h_{\mu_i-\lambda_j-i+j}[A_{\lambda_j}-B_{j-1}]\right)_{i,j=1}^n,$$
$$c_{\lambda,\mu}(\boldsymbol{\alpha},\boldsymbol{\beta}) = \det\left(h_{\lambda_i-\mu_j-i+j}[-A_{\lambda_i-1}+B_{i-1}]\right)_{i,j=1}^n.$$

Schur positivity + Combinatorial models I

$$G_\lambda(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta}) = \sum_{\mu\supseteq\lambda}\ \sum_{(E,T)\in\mathrm{IET}_\mathbb{Z}(\mu/\lambda)\times\mathrm{SSYT}(\mu)} \mathrm{wt}(E)\boldsymbol{x}^T,$$

where $\mathrm{wt}(E) = \prod_{(i,j)\in\mu/\lambda}(\alpha_{E(i,j)}-\beta_{E(i,j)-c(i,j)})$, and $c(i,j)=j-i$.

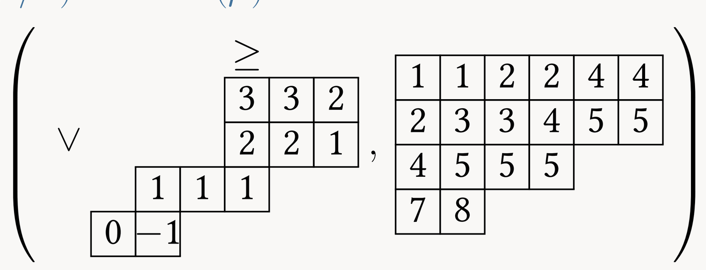
$$g_\lambda(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta}) = \sum_{\mu\subseteq\lambda}\ \sum_{(E,T)\in\mathrm{ET}_\mathbb{Z}(\lambda/\mu)\times\mathrm{SSYT}(\mu)} \mathrm{wt}'(E)\boldsymbol{x}^T,$$

where $\mathrm{wt}'(E) = \prod_{(i,j)\in\lambda/\mu}(-\alpha_{E(i,j)+c(i,j)}+\beta_{E(i,j)})$. Moreover, $G_\lambda(\boldsymbol{x};\boldsymbol{\alpha},-\boldsymbol{\beta})$ and $g_\lambda(\boldsymbol{x};-\boldsymbol{\alpha},\boldsymbol{\beta})$ are Schur-positive.

Inelegant tableaux reverse-$\mathrm{SSYT}_\mathbb{Z}$ with

$$\lambda_i \ge (\text{entries in row } i) > \min(\mu_i-i,0).$$

$\mathrm{IET}_\mathbb{Z}(\mu/\lambda)\times\mathrm{SSYT}(\mu)$

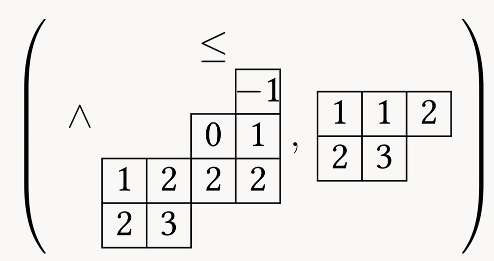


$$\mathrm{wt} = \alpha_3\alpha_3\alpha_2\alpha_2\alpha_2\alpha_1(\underline{\alpha_1}-\beta_2)(\alpha_1-\underline{\beta_1})\alpha_1 \times(-\beta_3)(-\beta_1)x_1^2x_2^3x_3^2x_4^4x_5^5x_7x_8$$

Elegant tableaux $\mathrm{SSYT}_\mathbb{Z}$ with

$$\min(i-\mu_i,1) \le (\text{entries in row } i) < i.$$

$\mathrm{ET}_\mathbb{Z}(\lambda/\mu)\times\mathrm{SSYT}(\mu)$



$$\mathrm{wt} = (-\alpha_2)(-\alpha_1)(-\alpha_3+\underline{\beta_1})\beta_1(\underline{-\alpha_1}+\beta_2) \times(-\alpha_2+\underline{\beta_2})(-\alpha_3+\underline{\beta_2})\beta_2(\underline{-\alpha_1}+\beta_3)x_1^2x_2^2x_3$$

Combinatorial models II

$$G_\lambda(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta}) = \sum_{T\in\mathrm{MMSVT}(\lambda)} \mathrm{wt}(T),$$

where $\mathrm{wt}(T) = \prod_{(i,j)\in\lambda}\boldsymbol{x}^{T(i,j)}\alpha_j^{\mathrm{unmarked}(T(i,j))-1}(-\beta_i)^{\mathrm{marked}(T(i,j))}$,

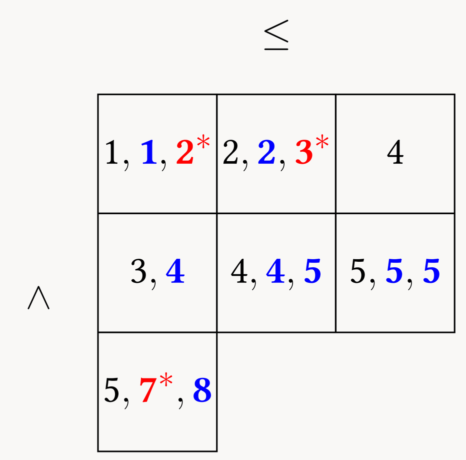
$$g_\lambda(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta}) = \sum_{T\in\mathrm{MRPP}(\lambda)} \mathrm{wt}'(T),$$

where $\mathrm{wt}'(T) = \prod_{(i,j)\in\lambda}\mathrm{wt}'(T(i,j))$, and

$$\mathrm{wt}'(T(i,j)) = \begin{cases} -\alpha_j & \text{if } T(i,j) \text{ is marked,}\\ \beta_{i-1} & \text{if } T(i,j) \text{ is not marked and } T(i,j)=T(i-1,j),\\ x_{T(i,j)} & \text{if } T(i,j) \text{ is not marked and } T(i,j)\ne T(i-1,j).\end{cases}$$

marked multiset-valued tableaux $\mathrm{MMSVT}(\lambda)$

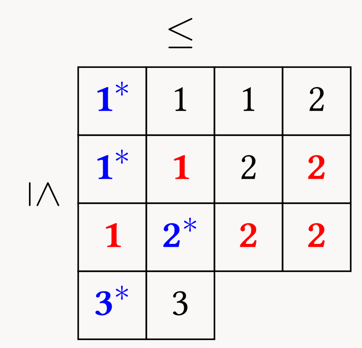
- entries: multisets $\{a_1\le a_2\le\dots\}$,
- a_i can be marked if $a_{i-1}<a_i$.



$$\mathrm{wt} = \alpha_1^3\alpha_2^3\alpha_3^2(-\beta_1)^2(-\beta_3)x_1^2x_2^3x_3^2x_4^4x_5^5x_7x_8$$

marked reverse plane partitions $\mathrm{MRPP}(\lambda)$

- entries: integers $T_{i,j}$,
- $T_{i,j}$ can be marked if $T_{i,j}=T_{i,j+1}$.



$$\mathrm{wt} = (-\alpha_1)^3(-\alpha_2)\beta_1^2\beta_2^3x_1^2x_2^2x_3$$

Duality with respect to the Hall inner product

$$\langle G_\lambda(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta}), g_\mu(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta})\rangle = \delta_{\lambda,\mu}.$$

Jacobi–Trudi-like formulas

Let λ and μ be partitions with at most n parts and $\mu\subseteq\lambda$. Let $\boldsymbol{r}=(r_1,\dots,r_n)$ and $\boldsymbol{s}=(s_1,\dots,s_n)$ be positive integer sequences of length n. Suppose that $r_i\le r_{i+1}$ and $s_i\le s_{i+1}$ whenever $\mu_i<\lambda_{i+1}$. Then

$$G_{\lambda/\mu}^{\mathrm{row}(\boldsymbol{r},\boldsymbol{s})}(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta}) = C\det\left(h_{\lambda_i-\mu_j-i+j}[X_{[r_j,s_i]}\ominus(A_{\lambda_i}-A_{\mu_j}-B_{i-1}+B_j)]\right)_{i,j=1}^n,$$
$$G_{\lambda'/\mu'}^{\mathrm{col}(\boldsymbol{r},\boldsymbol{s})}(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta}) = D\det\left(e_{\lambda_i-\mu_j-i+j}[X_{[r_j,s_i]}\ominus(A_{i-1}-A_j-B_{\lambda_i}+B_{\mu_j})]\right)_{i,j=1}^n,$$
$$g_{\lambda/\mu}^{\mathrm{row}(\boldsymbol{r},\boldsymbol{s})}(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta}) = \det\left(h_{\lambda_i-\mu_j-i+j}[X_{[r_j,s_i]}-A_{\lambda_i-1}+A_{\mu_j}+B_{i-1}-B_{j-1}]\right)_{i,j=1}^n,$$
$$g_{\lambda'/\mu'}^{\mathrm{col}(\boldsymbol{r},\boldsymbol{s})}(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta}) = \det\left(e_{\lambda_i-\mu_j-i+j}[X_{[r_j,s_i]}-A_{i-1}+A_{j-1}+B_{\lambda_i-1}-B_{\mu_j}]\right)_{i,j=1}^n,$$

where $X_{[r,s]} = x_r+x_{r+1}+\cdots+x_s$, and

$$C = \prod_{i=1}^n\prod_{l=r_i}^{s_i}(1-\beta_i x_l), \qquad D = \prod_{i=1}^n\prod_{l=r_i}^{s_i}(1-\alpha_i x_l)^{-1}.$$

Involution ω

$$\omega\left(G_{\lambda/\mu}(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta})\right) = G_{\lambda'/\mu'}(\boldsymbol{x};-\boldsymbol{\beta},-\boldsymbol{\alpha}),$$
$$\omega\left(g_{\lambda/\mu}(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta})\right) = g_{\lambda'/\mu'}(\boldsymbol{x};-\boldsymbol{\beta},-\boldsymbol{\alpha}).$$

Open problem. Find a bijective map from $\mathrm{MMSVT}(\lambda)$ to $\cup_\mu(\mathrm{IET}_\mathbb{Z}(\mu/\lambda)\times\mathrm{SSYT}(\mu))$, or from $\mathrm{MRPP}(\lambda)$ to $\cup_\mu(\mathrm{ET}_\mathbb{Z}(\lambda/\mu)\times\mathrm{SSYT}(\mu))$?

Jihyeug Jang
4242ab@gmail.com