Overview

- We introduce refined canonical stable Grothendieck polynomials and their duals.
- We give Jacobi-Trudi-like formulas, and combinatorial models, Schur expansions, Schur positivity, and dualities of them.

Grothendieck polynomials

- Grothendieck polynomials were introduced for studying the Grothendieck ring of vector bundles on a flag variety.
  - By Buch (2002)
  - Buch (2002) generalized several well-studied variations of Grothendieck polynomials:
    - f-stable
    - g-stable
    - h-stable
    - Z-stable
  - New operation \( N \) in Sect. 2.2

Notations

- Complete homogeneous symmetric function \( h_\lambda(x) = \sum_{\nu \leq \lambda} x_\nu \).
- Elementary symmetric function \( e_\lambda(x) = \sum_{\nu \leq \lambda} x_\nu \).
- Plücker substitutions
  - For any symmetric function \( f(x) \), \( f(z_1 + z_2 + \cdots + z_n) \)
- For any formal power series \( Y, Z \), \( h_{\lambda}(Y[Z]) = (-1)^{|\lambda|} h_{\lambda}(Y) Z^{|\lambda|} \)

New operation \( \oplus \)

- We define \( h_\lambda(Y[Z]) = \sum_{\rho \leq \lambda} h_{\rho}(Y) Z^{|\rho|} \).

Schur functions

Main definitions (Refined canonical stable Grothendieck polynomials)

Let \( \delta_k = \alpha_1 + \cdots + \alpha_k \) and \( B_k = \beta_1 + \cdots + \beta_k \).

\[ G_{\lambda}(x, \alpha, \beta) = \prod_{1 \leq j < k \leq \ell} (\beta_j - \alpha_k)^{i_{j,k}}(\beta_j - \alpha_k)^{1-i_{j,k}} \]

Remark

Let \( 0 \leq (0, 0, \ldots), 1 = (1, 1, \ldots), \alpha_0 = (\alpha_0, \alpha_0, \ldots) \) and \( B_0 = (\beta_0, \beta_0, \ldots) \). Our generalizations \( G_{\lambda}(x, \alpha, \beta) \) and \( g_{\lambda}(x, \alpha, \beta) \) generalize several well-studied variations of Grothendieck polynomials:

- f-stable
- g-stable
- h-stable
- Z-stable

Combinatorial models II

- \( \mathcal{G}_{\lambda}(x, \alpha, \beta) = \sum_{\pi \in \mathbb{P}(\lambda)} w(T) \]

Duality with respect to the Hall inner product

- \( \mathcal{G}_{\lambda}(x, \alpha, \beta) = \mathcal{G}_{\lambda}(x, \beta, \alpha) \)

Jacobi–Trudi-like formulas

- Let \( \lambda \) and \( \mu \) be partitions with at most \( n \) parts and \( \mu \subseteq \lambda \).
- \( \mathcal{G}_{\lambda}^{(\mu)}(x, \alpha, \beta) = \prod_{1 \leq j < k \leq \ell} (\beta_j - \alpha_k)^{i_{j,k}}(\beta_j - \alpha_k)^{1-i_{j,k}} \]

Involution \( \omega \)

- \( \omega \left( \mathcal{G}_{\lambda}(x, \alpha, \beta) \right) = \mathcal{G}_{\lambda}(x, \beta, \alpha) \)
- \( \omega \left( \mathcal{G}_{\lambda}^{(\mu)}(x, \alpha, \beta) \right) = \mathcal{G}_{\lambda}^{(\mu)}(x, \beta, \alpha) \)

Open problem. Find a bijection map from \( \text{MMSVT}(\lambda) \) to \( \text{UP}_{\lambda, \mu}(\Lambda(n) \times \text{SYT}(\mu)) \), or from \( \text{MRFP}(\lambda) \) to \( \text{UP}_{\lambda, \mu}(\Lambda(n) \times \text{SYT}(\mu)) \)?