Studying triangulations of even-dimensional cyclic polytopes via directed graphs

Nicholas Williams, Lancaster University, nicholas.williams@lancaster.ac.uk

Summary

Motivated by understanding higher-dimensional cluster structures, we study triangulations of even-dimensional cyclic polytopes by associating directed graphs to them, inspired by [OT12].

We prove two results, one characterising a certain class of triangulations in terms of their graphs, and another giving a criterion for when a certain muation operation called a bistellar flip can be performed.

Cyclic polytopes

In even dimensions, the *cyclic polytope* $C(n+2d+1, 2d)$ is the convex hull of the images of the points

$$\left\{\frac{i}{n+2d+1}2\pi\right\}_{1\leqslant i\leqslant n+2d+1}$$

on the curve in $\mathbb{R}^{2d}$ given by $t \mapsto$

$$(\cos t, \sin t, \cos 2t, \sin 2t, \ldots, \cos dt, \sin dt).$$

Triangulations of cyclic polytopes

A *triangulation* of $C(n+2d+1, 2d)$ is a subdivision of it into $2d$-simplices.

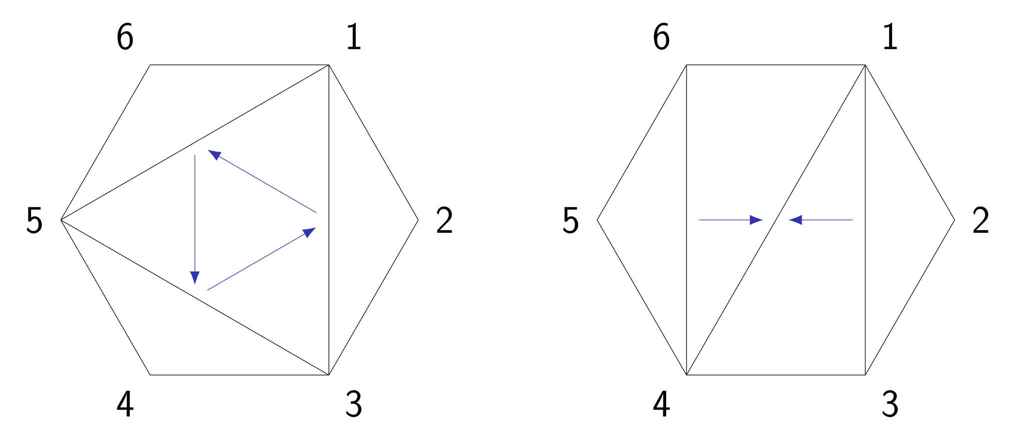
Just as 2D triangulations are determined by their diagonals, triangulations of $C(n+2d+1, 2d)$ are determined by their d-simplices which lie inside the polytope [OT12]. These are called *internal d-simplices*.

Bistellar flips

The higher-dimensional version of a flip of triangulations is a *bistellar flip*. They are given by replacing one internal d-simplex by another internal d-simplex.

For $d > 1$, not every internal d-simplex can be replaced using a bistellar flip, unlike for $d = 1$.

Figure 1: Triangulations of polygons with and without interior triangles



Directed graphs from triangulations

There is a well-known recipe from the theory of cluster algebras [FZ02] for associating a directed graph to a two-dimensional triangulation. The vertices of the graph are the diagonals of the triangulation, with an arrow $A \to B$ if B is clockwise from A in a triangle T.

For a triangulation of an arbitrary cyclic polytope, one can define the graph as follows.

The vertices of the graph are the internal d-simplices of the triangulation.

Then there are arrows $(a_0, a_1, \ldots, a_d) \to (a_0, \ldots, a_{i-1}, a_i + r, a_{i+1}, \ldots, a_d)$ between internal d-simplices (where r is minimal).

Figure 2: Constructing the directed graph for a triangulation of $C(8, 4)$

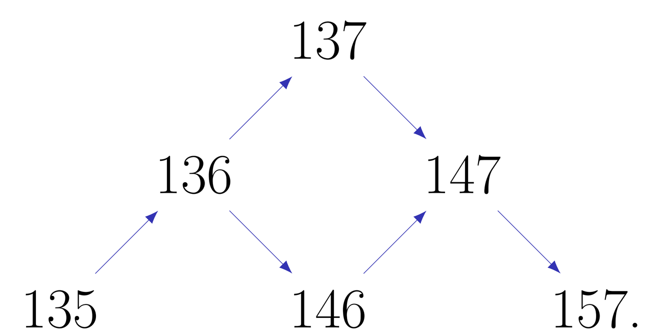
4-simplices of triangulation: $\{12345, 12356, 12367, 12378, 13456, 13467, 13478, 14567, 14578, 15678\}$

↓

Internal 2-simplices: $\{135, 136, 137, 146, 147, 157\}$

↓

Directed graph



Triangulations with no interior $(d+1)$-simplices

The graph of a triangulation of $C(n+2d+1, 2d)$ detects whether or not there are interior $(d+1)$-simplices, i.e., one whose facets are all internal d-simplices (Figure 1).

Theorem ([Wil])

A triangulation of $C(n+2d+1, 2d)$ has no interior $(d+1)$-simplices iff its directed graph is a cut.

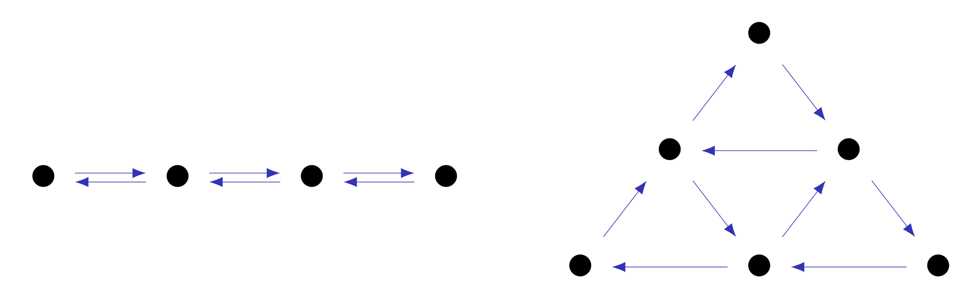
We do not define cuts explicitly here. They are obtained from the directed graphs in Figure 3 by removing exactly one arrow from each cycle.

Corollary ([Wil])

Triangulations of $C(n+2d+1, 2d)$ with no interior $(d+1)$-simplices form a connected subgraph of the flip graph of $C(n+2d+1, 2d)$.

Here the *flip graph* of a polytope is the undirected graph with triangulations of the polytope as vertices and bistellar flips as edges.

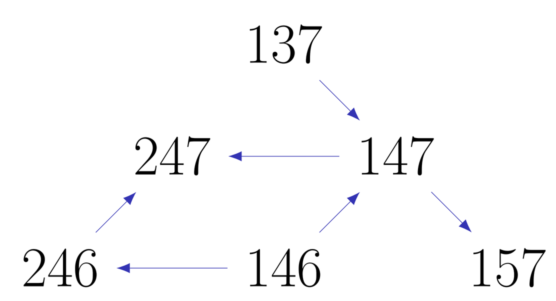
Figure 3: Cuts are obtained by removing exactly one arrow from each $(d+1)$-cycle of graphs such as the following



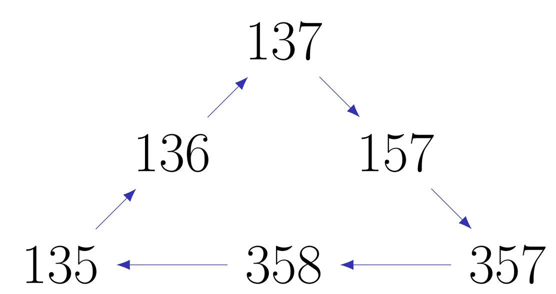
$d = 1, n = 4$ $\quad$ $d = 2, n = 3$

Figure 4: Triangulations of $C(8, 4)$ with and without interior 3-simplices

Without an interior 3-simplex



With an interior 3-simplex



Criterion for bistellar flips

Secondly, the graph of a triangulation allows one to detect where bistellar flips can be performed.

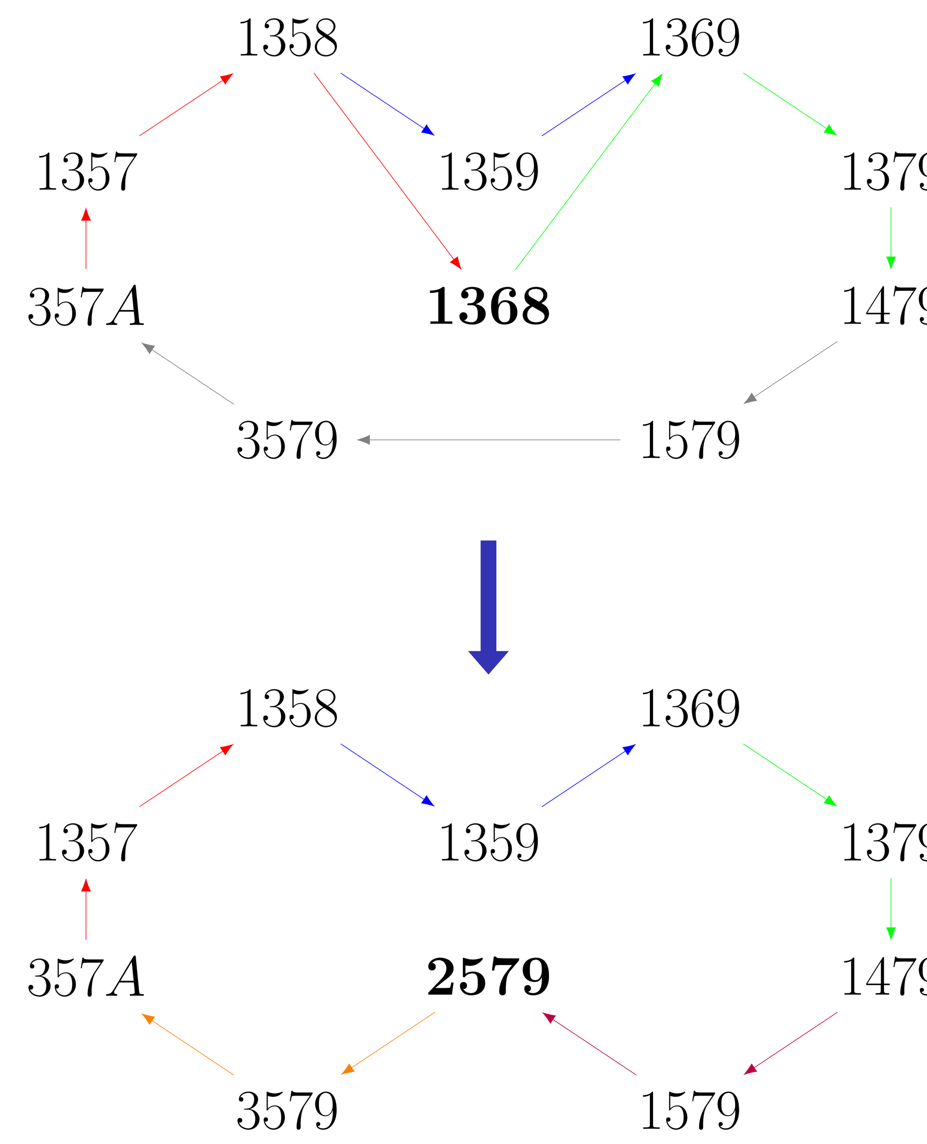
We show that the directed graph of a triangulation can be decomposed into certain paths which we call *retrograde paths*.

Theorem ([Wil])

A internal d-simplex can be replaced in a bistellar flip iff it does not occur in the middle of a retrograde path.

In Figure 5 we illustrate the retrograde paths in the directed graph by drawing each retrograde path in a single colour.

Figure 5: Performing a bistellar flip at 1368



References

[FZ02] Sergey Fomin and Andrei Zelevinsky. "Cluster algebras. I. Foundations". In: *J. Amer. Math. Soc.* 15.2 (2002).

[OT12] Steffen Oppermann and Hugh Thomas. "Higher-dimensional cluster combinatorics and representation theory". In: *J. Eur. Math. Soc. (JEMS)* 14.6 (2012).

[Wil] Nicholas J. Williams. *Quiver combinatorics and triangulations of cyclic polytopes.* To appear in *Algebraic Combinatorics*. arXiv: 2112.09189 [math.CO].