Quaisymmetric harmonics of the exterior algebra

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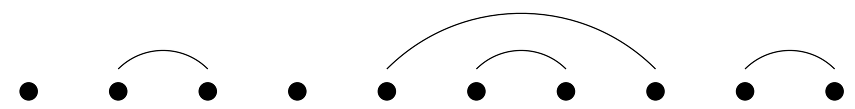
extended abstract

Overview



What do the *coinvariants* look like in the setting of the exterior algebra with a *quasisymmetric* action of $\mathbb{S}_n$? The short answer: there is a basis indexed by $\{0,1\}$-ballot sequences similar to the quasisymmetric coinvariants in the commutative polynomial ring.

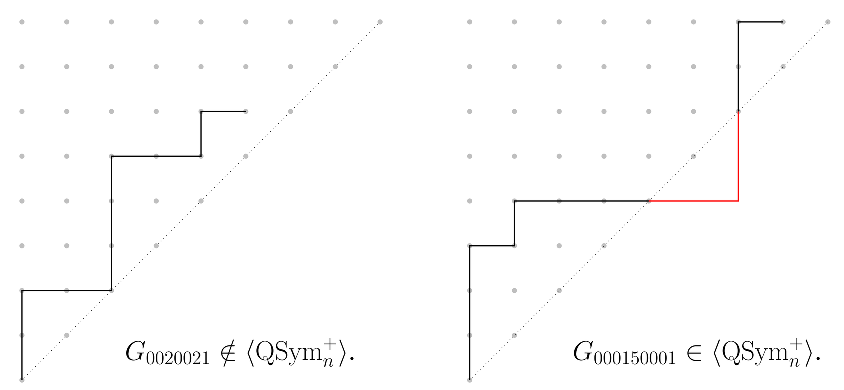
$$\mathrm{Hilb}_{\mathbb{Q}[\theta_1,\dots,\theta_n]/\langle \mathrm{QSym}_n^+\rangle}(q) = \sum_{k=0}^{\lfloor n/2\rfloor} f^{(n-k,k)} q^k.$$

The History

The ring of quasisymmetric function QSym_n includes the ring of symmetric function. Many combinatorial structures of QSym_n parallel that of Sym_n. Hivert described an action of the Temperley-Lieb algebra TL_n, thought of as a quotient of the symmetric group algebra, on $\mathbb{Q}[x_1,\dots,x_n]$ making QSym_n an isotypic trivial representation of TL_n. In 2003, Aval, F. Bergeron, and N. Bergeron studied QSym coinvariant spaces

$$\mathbb{Q}[x_1,\dots,x_n]/\langle \mathrm{QSym}_n^+\rangle$$

obtained by replacing the ideal of symmetric functions with no constant terms with the ideal of quasisymmetric functions with no constant terms. They developed a basis $\{G_\pi\}$ of the polynomial ring indexed by lattice paths and gave a criterion for membership of $\langle \mathrm{QSym}_n^+\rangle$ by whether the indexing path crosses the diagonal.



Surprisingly, the dimensions of the QSym coinvariants are equal to the Catalan numbers!

The Setup

Motivated by recent development in coinvariant theory involving fermionic (anticommuting) variables, we study the QSym coinvariant spaces of the exterior algebra (Exterior Quasisymmetric Coinvariants)

$$EQC_n := \mathbb{Q}[\theta_1,\dots,\theta_n]/\langle \mathrm{QSym}_n^+\rangle.$$

- The exterior algebra is $\mathbb{Q}[\theta_1,\dots,\theta_n]$ with $\theta_i\theta_j = -\theta_j\theta_i$.
- Index monomials by sets $\theta_{\{1,3,5\}} = \theta_1\theta_3\theta_5$.
- Extend Hivert's $\mathbb{S}_n$-action: Act by permutation but ignore signs

$$(2,5)\cdot\theta_{\{1,3,5\}} = \theta_{(2,5)\{1,3,5\}} = \theta_{\{1,2,3\}} = \theta_1\theta_2\theta_3.$$

- Invariant polynomials are called *quasisymmetric*. A basis consists of

$$F_{1^k} = \sum_{\substack{A\subseteq[n]\\|A|=k}} \theta_A, \quad k = 1,\dots,n.$$

In $\mathbb{Q}[\theta_1,\theta_2,\theta_3]$,

$$F_1 = \theta_1+\theta_2+\theta_3,$$
$$F_{11} = \theta_1\theta_2+\theta_1\theta_3+\theta_2\theta_3,$$
$$F_{111} = \theta_1\theta_2\theta_3$$

- The quasisymmetric invariant ideal is $\langle \mathrm{QSym}_n^+\rangle = \langle F_1,\dots,F_{1^n}\rangle$.

Detour to symmetric invariants

Act by permutations but with signs.

$$(2,5)\theta_{\{1,3,5\}} = (2,5)\theta_1\theta_3\theta_5 = \theta_1\theta_3\theta_2 = -\theta_1\theta_2\theta_3 = -\theta_{\{1,2,4\}}.$$

The symmetric invariants $\mathbb{Q}\{\theta_1+\cdots+\theta_n\}$ form a 1-dimensional subspace.

$$(1+(1,2))\cdot\theta_1\theta_2 = \theta_1\theta_2 - \theta_1\theta_2 = 0.$$

A parallel to the classic coinvariant story:

Theorem. The exterior algebra $\mathbb{Q}[\theta_1,\dots,\theta_n]$ is free over its *symmetric* polynomials.

The structure of $\langle \mathrm{QSym}_n^+\rangle$

We extend a result of Fishel, Lapointe and Pinto by determining the structural coefficients of QSym_n.

Theorem. The algebra $\mathrm{QSym}_n = \mathbb{Q}[F_1,\dots,F_{1^n}]$ is commutative and

$$F_{1^r}F_{1^s} = \begin{cases} \binom{\lfloor \frac{r+s}{2}\rfloor}{\lfloor \frac{r}{2}\rfloor} F_{1^{r+s}} & \text{if } rs\equiv 0 \pmod 2 \\ 0 & \text{otherwise}\end{cases}.$$

That means the ideal $\langle \mathrm{QSym}_n^+\rangle$ is very simple!

$$\langle \mathrm{QSym}_n^+\rangle = \langle F_1, F_{11}\rangle. \qquad (1)$$

Harmonics in the exterior algebra

Let $\mathbb{Q}[\theta_1,\dots,\theta_n]$ act on itself by partial differentiation. Differential operators ∂_{θ_k} also anticommute.

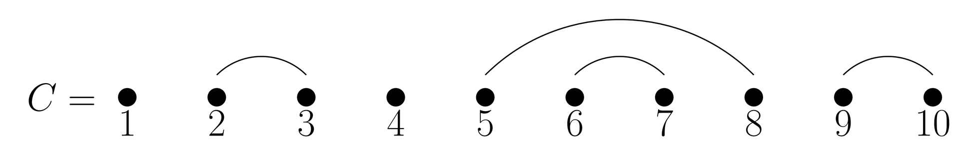
$$\partial_{\theta_1}\theta_1\theta_2 = \theta_2 \quad\text{and}\quad \partial_{\theta_2}\theta_1\theta_2 = \theta_1(-\partial_{\theta_2})\theta_2 = -\theta_1.$$

The harmonics (Exterior Quasisymmetric Harmonics) are

$$EQH_n := \left\{ q\in\mathbb{Q}[\theta_1,\dots,\theta_n] : \sum_{1\le i\le n}\partial_{\theta_i} q = 0 \quad\text{and}\quad \sum_{1\le i<j\le n}\partial_{\theta_j}\partial_{\theta_i} q = 0\right\}.$$

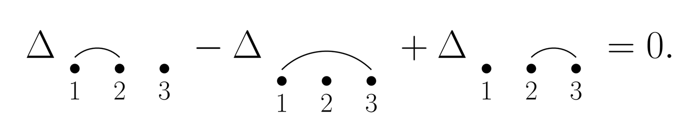
And $EQC_n \cong EQH_n$.

Non-crossing pairings build harmonic polynomials.



$$\Delta_C = (\theta_3-\theta_2)(\theta_7-\theta_6)(\theta_8-\theta_5)(\theta_{10}-\theta_9).$$

However, they are not linearly independent:



Ballot sequences and non-crossing pairing

A $\{0,1\}$-sequence is ballot (aka Yamanouchi) if every prefix has at least as many 0's as 1's.

Yes ballot: 0010001101. Not ballot: 0110001101.

From ballot sequence to non-crossing pairings.

$\alpha =$ 0 0 1 0 0 0 1 1 0 1

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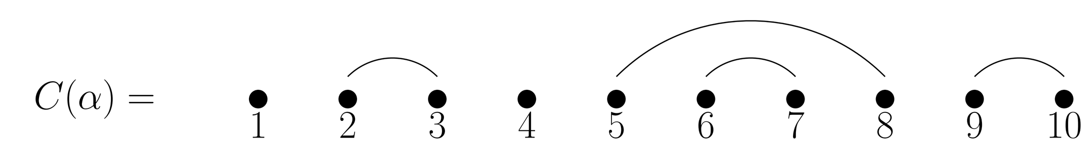
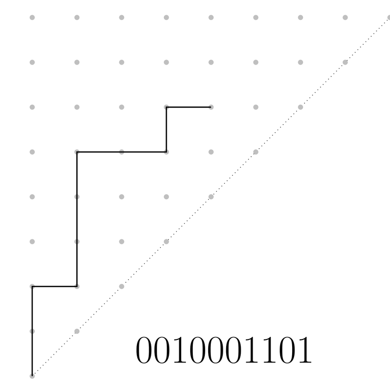


Interpret a 0 as a North step and a 1 as an East step:



Theorem. $\{\Delta_{C(\alpha)} : \alpha \text{ is a ballot sequence }\}$ is linearly independent in EQH_n.

A linear basis of $\mathbb{Q}[\theta_1,\dots,\theta_n]$ indexed by $\{0,1\}$-sequences

Recursively define a family of polynomials analogous to that of Aval-Bergeron-Bergeron.

$$G_{1^s0^{n-s}} = F_{1^s}$$
$$G_{u\mathbf{0}1^s0^{n-k-s}} = G_{u1^s0^{n-k-s}\mathbf{0}} - (-1)^{\#\text{ of 1's in }u}\theta_k G_{u1^{s-1}0^{n-k-s+1}\mathbf{0}},$$

where $k=\ell(u)+1$ is the position of the right-most non-trailing 0.

The idea is to recursively get "closer" to F_{1^k}. Examples:

- Indexed by a non-ballot sequence:

$$\begin{aligned} G_{01\mathbf{0}110} &= G_{01110\mathbf{0}} - (-1)^1\theta_3 G_{01100\mathbf{0}} \\ &= (G_{111000} - \theta_1 G_{110000}) + \theta_3(G_{110000} - \theta_1 G_{100000}) \\ &= \theta_2\theta_4\theta_5 + \theta_2\theta_4\theta_6 + \theta_2\theta_5\theta_6 + 2\theta_3\theta_4\theta_5 + 2\theta_3\theta_4\theta_6 + 2\theta_3\theta_5\theta_6 + \theta_4\theta_5\theta_6.\end{aligned}$$

- Indexed by a ballot sequence:

$$\begin{aligned} G_{0\mathbf{0}1100} &= G_{01100\mathbf{0}} - \theta_2 G_{01000\mathbf{0}} \\ &= (G_{110000} - \theta_1 G_{100000}) - \theta_2(G_{100000} - \theta_1 G_{000000}) \\ &= \theta_3\theta_4 + \theta_3\theta_5 + \theta_3\theta_6 + \theta_4\theta_5 + \theta_4\theta_6 + \theta_5\theta_6.\end{aligned}$$

Theorem.

1. The lexicographical leading term of G_α is θ^α.
2. The set $\{G_\alpha : \alpha \text{ is not a ballot sequence }\}$ is a basis for $\langle \mathrm{QSym}_n^+\rangle$.
3. The set $\{G_\alpha : \alpha \text{ is a ballot sequence }\}$ is a basis for EQC_n.

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