Hook length bias in odd versus distinct partitions

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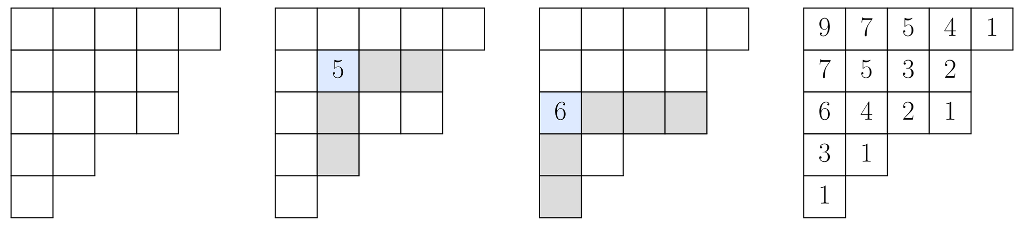
Background

Partitions, Young diagrams and hook lengths

A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_j)$ of size $n \in \mathbb{N}_0$ is a non-increasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_j$ called parts that add up to n. For example, for $n = 5$, the list of partitions are: $(5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1), (1,1,1,1,1)$.

Each partition is naturally equipped with a Young diagram, a left-justified vertical array of boxes with rows corresponding to parts. Each box in a Young diagram of λ may be labeled with a hook number, also called hook length, which, informally, is the number of boxes in the upside-down-L-shaped portion of the diagram with the box appearing as its corner. For example, the partition $\lambda = (5,4,4,2,1)$ of size 16 has the following Young diagram, with the hook length of each box marked.



Euler's identity and Beck's conjecture

Our work involves the following two types of partitions: partition into odd parts (odd partitions) and partitions into distinct parts (distinct partitions). We denote by $\mathcal{O}(n)$, respectively $\mathcal{D}(n)$, the set of odd, respectively distinct, partitions of n.

- Euler's identity [1] states that $|\mathcal{O}(n)| = |\mathcal{D}(n)|$ for all $n \geq 0$.
- Euler's identity can be proved combinatorially via the Glaisher's bijection [5] $\mathcal{D}(n) \to \mathcal{O}(n)$, which splits each even part in a distinct partition into two equal parts recursively until all parts are odd.
- Furthermore, Beck [6] conjectured and Andrews [2] proved:

  *The total number of parts in all odd partitions of n minus the total number of parts in all distinct partitions of n is equal to **the number of partitions of n where exactly one part size is even**.*

For example, when $n = 5$, we have

	odd partitions	distinct partitions	partitions with one even part size
	(5) $(3,1,1)$ $(1,1,1,1,1)$	(5) $(3,2)$ $(4,1)$	$(4,1)$ $(3,2)$ $(2,2,1)$ $(2,1,1,1)$
count	3	3	$4(= \mathbf{9} - \mathbf{5})$
number of parts	**9**	**5**	-

Our question

Instead of tracking the **total number of parts**, we are interested in tracking the **total number of hooks of a given length** in odd versus distinct partitions.

For a positive integer t, let $a_t(n)$ (respectively $b_t(n)$) be the total number of hooks of length t in all odd (respectively distinct) partitions of n.

Question: How does $a_t(n)$ compare with $b_t(n)$ for a fixed t? Is one always greater than or equal to the other (for sufficiently large n)? If so, does the difference count some other combinatorial data?

Previous result: the $t = 1$ case

- In any given partition, the number of hooks of length 1 is precisely the number of different part sizes. (See the Young diagrams above for an example.)
- Beck [7] conjectured and Andrews [2] proved the following result.

  *The total number of parts in all distinct partitions of n minus the total number of different part sizes in all odd partitions of n is equal to $c(n)$, **the number of partitions of n with exactly one part occurring three times while all other parts occur only once**.*
- In other words, for $n \geq 0$, $b_1(n) - a_1(n) = c(n) \geq 0$. **This answers our question completely for the $t = 1$ case.**

Our Conjecture

For $t = 1$ we showed in the previous block that $b_1(n) \geq a_1(n)$. On the other hand, Euler's identity yields $\sum_{t \geq 1} a_t(n) = n|\mathcal{O}(n)| = n|\mathcal{D}(n)| = \sum_{t \geq 1} b_t(n)$, which indicates that the hook length bias reverses (compared to the $t = 1$ case) for some $t \geq 2$. In fact, we conjecture such reversal for all $t \geq 2$ for large enough n. More specifically:

Conjecture. *For every integer $t \geq 2$, there exists an integer N_t such that for all $n > N_t$, we have $a_t(n) \geq b_t(n)$, and $a_t(n) - b_t(n) \to \infty$ as $n \to \infty$. Moreover, for $2 \leq t \leq 10$, we conjecture the following values of N_t:*

t	2	3	4	5	6	7	8	9	10
N_t	0	7	8	18	16	34	34	56	59

Data supporting this conjecture was obtained by enumerating partitions and not from generating functions; this is because the generating functions for $a_t(n)$ and $b_t(n)$ are difficult to derive explicitly. We are, however, able to write down generating functions for $t = 2$ and $t = 3$, which we use to prove the conjecture for $t = 2$ and $t = 3$.

Main Results and outline of proof

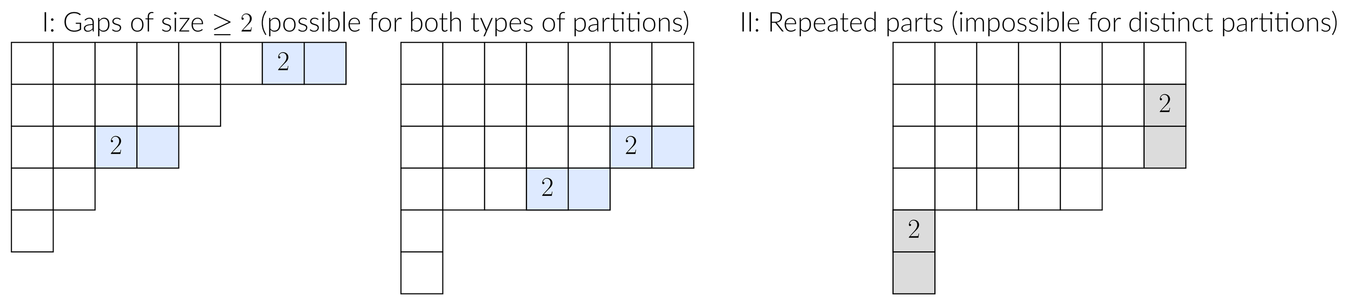
Theorem 1. *The above conjecture is true for $t = 2$ and $t = 3$ with $N_2 = 0$ and $N_3 = 7$, respectively.*

Proof outline and remarks.

- We first find the generating functions $a_2(n), b_2(n), a_3(n), b_3(n)$.
- The generating function for $a_2(n) - b_2(n)$ leads to a new combinatorial interpretation of $a_2(n) - b_2(n)$ (see the right column), which will be evidently non-negative, showing $a_2(n) \geq b_2(n)$ for all $n \geq 0$.
- *The previous two steps were demonstrated in detail in [3] and will be the focus of the rest of this poster.*
- From the generating functions, to show that $a_3(n) \geq b_3(n)$ for all $n > 7$, and to show that $a_t(n) - b_t(n) \to \infty$ as $n \to \infty$ for $t = 2, 3$, we need additional tools from analytic number theory, in particular the circle method. This was outlined in [3] and completed in [4], and we will not focus on this part in the poster.

The $t = 2$ case

There are precisely two ways in which hooks of length 2 can occur in a Young diagram:



Let $h_t(\lambda)$ denote the number of hooks of length t in a partition λ. Let

$$a_t(m,n) := \#\{\lambda \in \mathcal{O}(n) : h_t(\lambda) = m\} \quad \text{and} \quad b_t(m,n) := \#\{\lambda \in \mathcal{D}(n) : h_t(\lambda) = m\}.$$

Odd Partitions. In an odd partition, the part 1 contributes one hook of length 2 if and only if it occurs at least twice; a part ≥ 3 contributes one hook of length 2 if occurring only once, two if occurring at least twice. We have

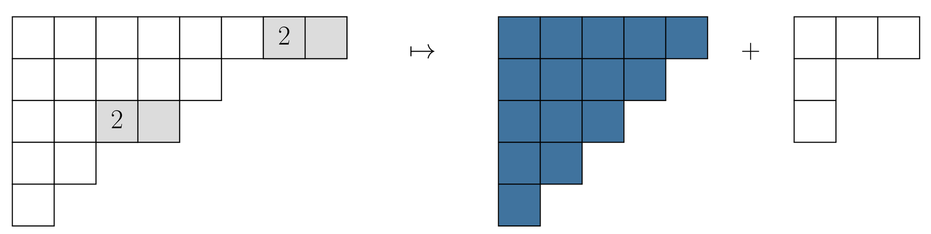
$$\sum_{n,m \geq 0} a_2(m,n) z^m q^n = \left(1 + q + \frac{zq^2}{1-q}\right) \prod_{n=1}^{\infty} \left(1 + zq^{2n+1} + \frac{z^2 q^{2(2n+1)}}{1 - q^{2n+1}}\right),$$

where $\left(1 + q + \frac{zq^2}{1-q}\right) = 1 + q + zq^2 + zq^3 + \cdots$ keeps track of parts of size 1, and $\left(1 + zq^{2n+1} + \frac{z^2 q^{2(2n+1)}}{1-q^{2n+1}}\right) = 1 + zq^{2n+1} + z^2 q^{2(2n+1)} + z^2 q^{3(2n+1)} + \cdots$ keeps track of parts of size $2n+1 \geq 3$. Note that $a_2(n) = \sum_{m \geq 0} m \cdot a_2(m,n)$, therefore

$$\sum_{n \geq 0} a_2(n) q^n = \frac{\partial}{\partial z} \left(\sum_{n,m \geq 0} a_2(m,n) z^m q^n \right) \Bigg|_{z=1} = \frac{1}{(q;q^2)_\infty} \left(q^2 + \sum_{n \geq 1} \left(q^{2n+1} + q^{2(2n+1)} \right) \right).$$

Here and throughout, the *q-Pochhammer symbol* is defined for $n \in \mathbb{N}_0 \cup \{\infty\}$ by $(a;q)_n := \prod_{j=0}^{n-1} (1 - aq^j)$.

Distinct Partitions. In a distinct partition, the number of hooks of length 2 equals the number of gaps of size ≥ 2, which, after removing a staircase from the Young diagram, is the number of different part sizes in what remains.



Therefore, $\sum_{n,m \geq 0} b_2(m,n) z^m q^n = \sum_{n \geq 1} q^{n(n+1)/2} \prod_{j=1}^{n} \left(1 + \frac{zq^j}{1-q^j}\right)$.

Then, differentiating with respect to z, evaluating at $z = 1$, and using some well-known q-series identities, we obtain

$$\sum_{n \geq 0} b_2(n) q^n = \frac{\partial}{\partial z} \left(\sum_{n,m \geq 0} b_2(m,n) z^m q^n \right) \Bigg|_{z=1} = \frac{q^2}{1-q} (-q^2;q)_\infty.$$

Hook bias for $t = 2$. After simplifications we get

$$\sum_{n \geq 0} (a_2(n) - b_2(n)) q^n = \frac{q^3}{1-q^2} \left(\left(-q^3;q\right)_\infty + q^{1+2} \left(-q^3;q\right)_\infty \right),$$

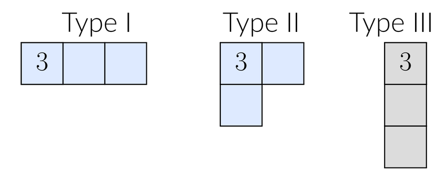
which clearly has non-negative coefficients. To see a combinatorial interpretation:

- $(-q^3;q)_\infty = (1+q^3)(1+q^4)(1+q^5)\cdots$ is the generating function for the number of distinct partitions of n that do not have 1 and 2 as parts.
  - Equivalently (via Glaisher's bijection), odd partitions of n where 1 has multiplicity $0 \pmod 4$.
- $q^{1+2}(-q^3;q)_\infty$ is the generating function for the number of distinct partitions of n that have both 1 and 2 as parts.
  - Equivalently (via Glaisher's bijection), odd partitions of n where 1 has multiplicity $3 \pmod 4$.
- $q^3/(1-q^2) = q^3 + q^5 + \cdots$ is the generating function of the number of partitions of n consisting of a single odd part ≥ 3.

Therefore, $a_2(n) - b_2(n)$ is **the number of odd partitions of n where the part 1 has multiplicity 0 or $3 \pmod 4$ and one part ≥ 3 is marked.**

The $t = 3$ case, how we obtain generating functions

There are three types of hooks of length 3 that can occur in the partition.



In an odd partition, the part 1, if occurring at least three times, contributes one hook of Type III. A part ≥ 3 contributes one hook of Type I, unless it has a gap exactly 2 with the part below; a second occurrence of a part ≥ 3 contributes one hook of Type II while a third occurrence contributes one hook of Type III. **In a distinct partition**, each part ≥ 2 contributes one hook of length 3 unless it has a gap exactly 2 with the part below. (If the gap ≥ 3, then we have a hook of Type I; if the gap is 1, then Type II.)

We then construct the bivariate generating functions, differentiate, and evaluate at $z = 1$ to obtain the generating functions for $a_3(n)$ and $b_3(n)$, similar to the $t = 2$ case.

Further results and future directions

In the process of proving the $t = 3$ case, we [4] proved the following general linear inequalities for partitions into distinct parts, which are of independent interest:

Theorem 2. *Let $q(n) = |\mathcal{D}(n)|$. Suppose $\sum_{k=1}^{r} \alpha_k < \sum_{\ell=1}^{s} \beta_\ell$ where α_k's and β_ℓ's are positive integers. Then given any $\{\mu_k\}_{k=1}^{r} \in \mathbb{N}_0^r$, and $\{\nu_\ell\}_{\ell=1}^{s} \in \mathbb{N}_0^s$, there exists an N (explicitly depending on the given sequences) such that for $n > N$,*

$$\sum_{k=1}^{r} \alpha_k q(n + \mu_k) \leq \sum_{\ell=1}^{s} \beta_\ell q(n + \nu_\ell).$$

Future Problems: It would be nice to give an evidently-nonnegative combinatorial interpretation of $a_3(n) - b_3(n)$ for $n > 7$. Also, our conjecture in the $t \geq 4$ cases is widely open.

References

[1] George E. Andrews. *The theory of partitions*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1998. Reprint of the 1976 original.

[2] George E. Andrews. Euler's partition identity and two problems of George Beck. *Math. Student*, 86(1-2):115–119, 2017.

[3] Cristina Ballantine, Hannah Burson, William Craig, Amanda Folsom, and Boya Wen. Hook length bias in odd versus distinct partitions. *Seminaire Lotharingien de Combinatoire, Proceedings of the 35th International Conference on Formal Power Series and Algebraic Combinatorics*, 2023.

[4] Cristina Ballantine, Hannah Burson, William Craig, Amanda Folsom, and Boya Wen. Hook length biases and general linear partition inequalities. *arXiv preprint arXiv:2303.16512*, 2023.

[5] James Whitbread Lee Glaisher. A theorem in partitions. *Messenger of Math.*, 12:158–170, 1883.

[6] OEIS Foundation. Entry A090867 in The On-Line Encyclopedia of Integer Sequences. http://oeis.org/A090867, 2017.

[7] OEIS Foundation. Entry A265251 in The On-Line Encyclopedia of Integer Sequences. http://oeis.org/A265251, 2017.