

# Horizontal-strip LLT polynomials

Foster Tom

MIT

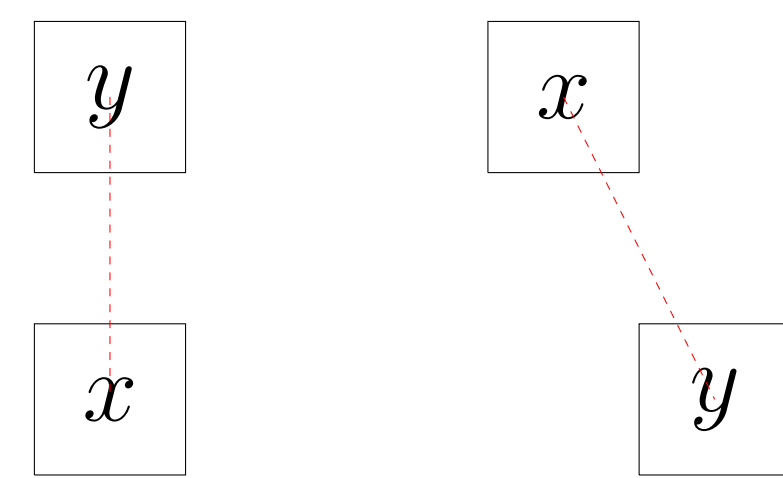
## Background

$\lambda$ : sequence of rows

$T$ : fill with weakly increasing positive integers

$x^T$ : monomial  $x_1^{\text{number of 1's}} x_2^{\text{number of 2's}} \dots$

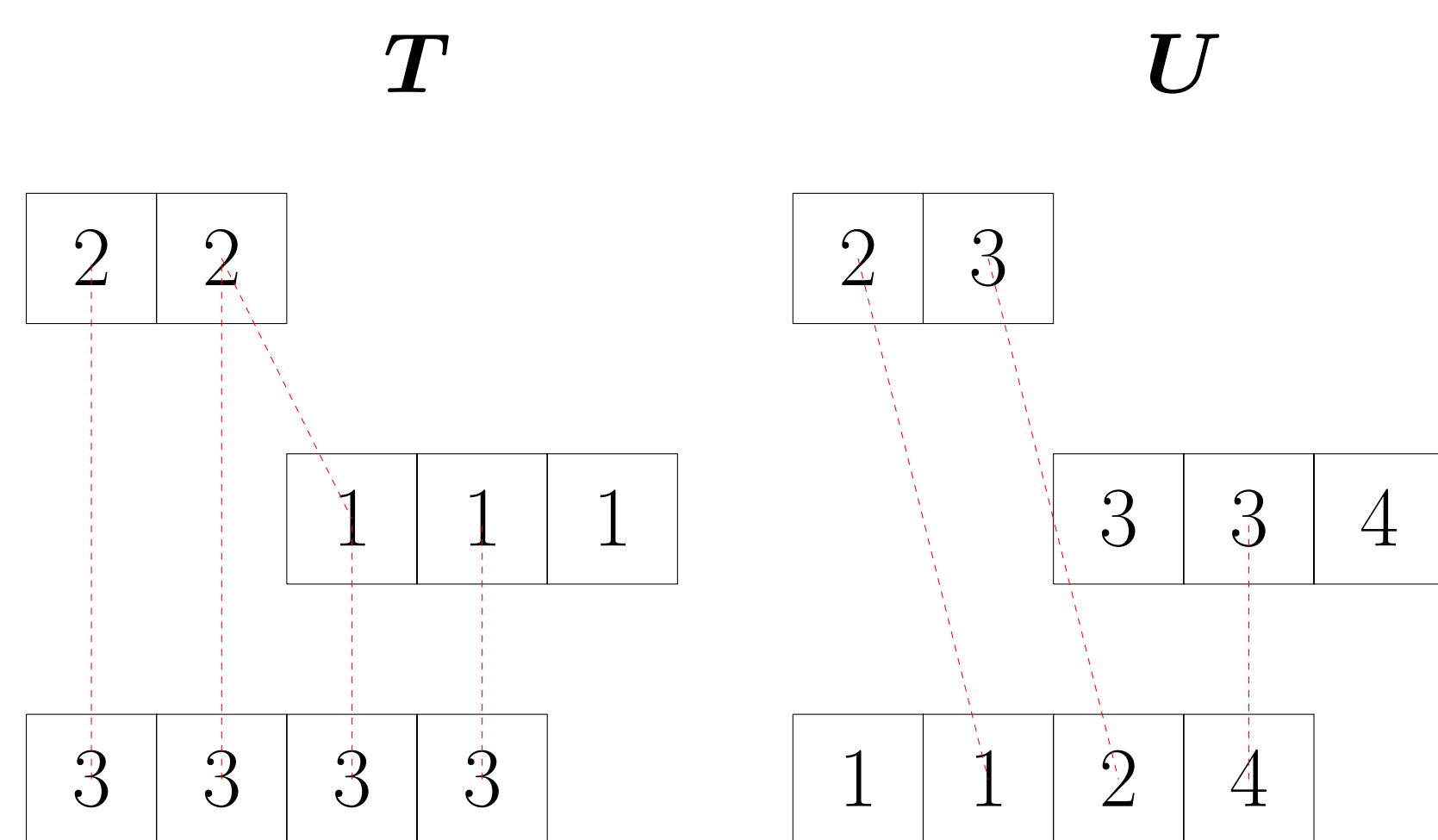
$\text{inv}(T)$ : number of pairs below with  $x > y$



**Horizontal-strip LLT polynomial:**

$$G_\lambda(x; q) = \sum_{T \in \text{SSYT}_\lambda} q^{\text{inv}(T)} x^T$$

**Example:**



$$\begin{aligned} G_\lambda(x; q) &= \dots + q^5 x_1^3 x_2^2 x_3^4 + \dots + q^3 x_1^2 x_2^2 x_3^3 x_4^2 + \dots \\ &= q^5 s_{432} + q^5 s_{441} + q^5 s_{522} + (q^5 + q^4) s_{531} \\ &\quad + 2q^4 s_{54} + 2q^4 s_{621} + (q^4 + 2q^3) s_{63} + q^3 s_{711} \\ &\quad + (2q^3 + q^2) s_{72} + (q^2 + q) s_{81} + s_9. \end{aligned}$$

**Theorem (Lascoux, Leclerc, Thibon 1997):**

$G_\lambda(x; q)$  is a symmetric function.

**Theorem (Grojnowski, Haiman 2007):**

$G_\lambda(x; q)$  is Schur-positive.

**Theorem (Grojnowski, Haiman 2007):** If the rows of  $\lambda$  are nested, then

$$G_\lambda(x; q) = \tilde{H}_\lambda(x; q) = \sum_{T \in \text{SSYT}(\lambda)} q^{\text{cocharge}(T)} s_{\text{shape}(T)}.$$

**Theorem (Carlsson, Mellit 2005):** We have

$$\frac{G_\lambda([x(q-1)]; q)}{(q-1)^n} = X_{\Gamma(\lambda)}(x; q),$$

the chromatic quasisymmetric function of a certain graph  $\Gamma(\lambda)$ .

## Main results

**Definition** For rows  $R$  and  $R'$ , define the integer

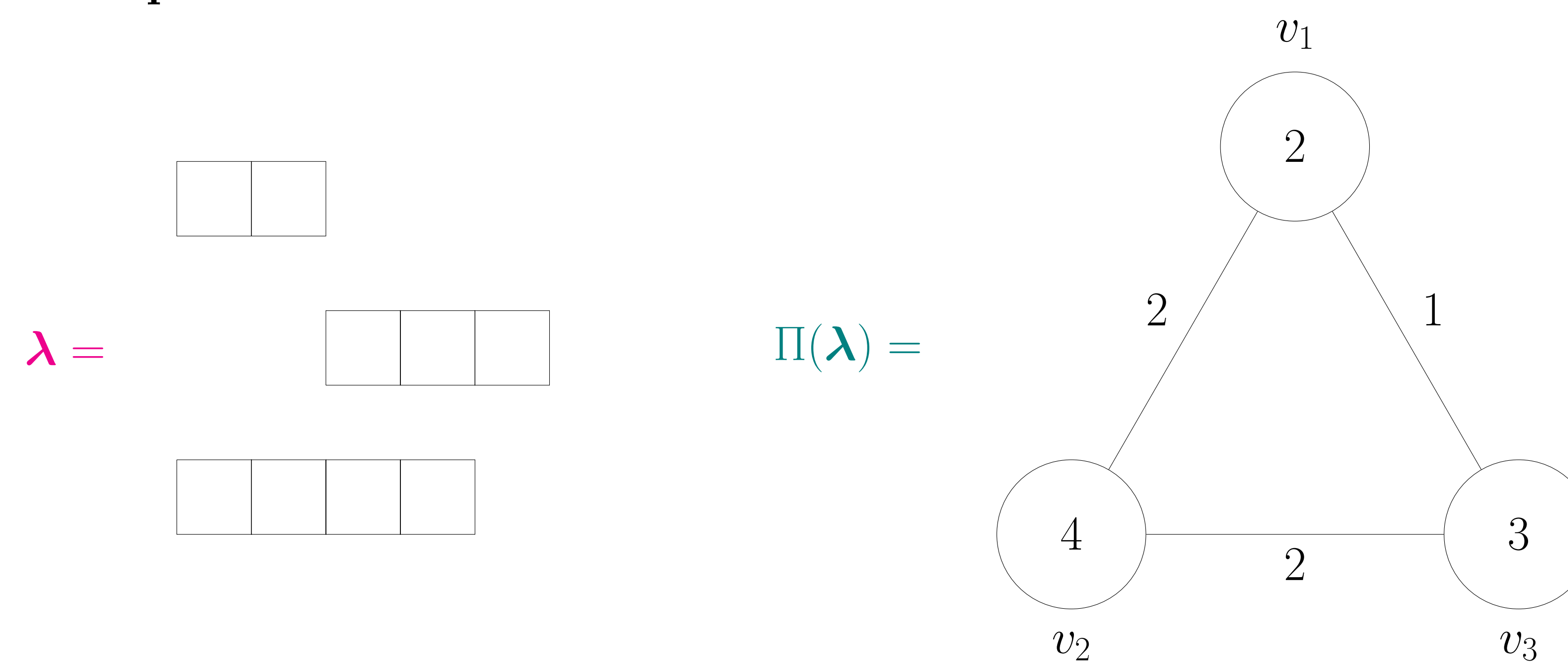
$$M(R, R') = \begin{cases} |R \cap R'| & \text{if } R \text{ starts weakly left of } R', \\ |R \cap R'^+| & \text{if } R \text{ starts strictly right of } R', \end{cases}$$

where  $R'^+$  is the row  $R'$  moved to the right one unit.

**Definition: Weighted graph  $\Pi(\lambda)$  associated to  $\lambda$**

- **Vertices:** rows of  $\lambda$
- **Weight:** number of cells in row
- **Edges:** join attacking rows
- **Weight:**  $M(R, R')$

**Example**



**Theorem: LLT determined by weighted graph**

$$\text{If } \Pi(\lambda) \cong \Pi(\mu), \text{ then } G_\lambda(x; q) = G_\mu(x; q).$$

**Theorem: A combinatorial Schur expansion**

$$\text{If } \Pi(\lambda) \text{ is a tree, then } G_\lambda(x; q) = \sum_{T \in \text{SSYT}(\alpha)} q^{\text{cocharge}_\Pi(T)} s_{\text{shape}(T)}$$

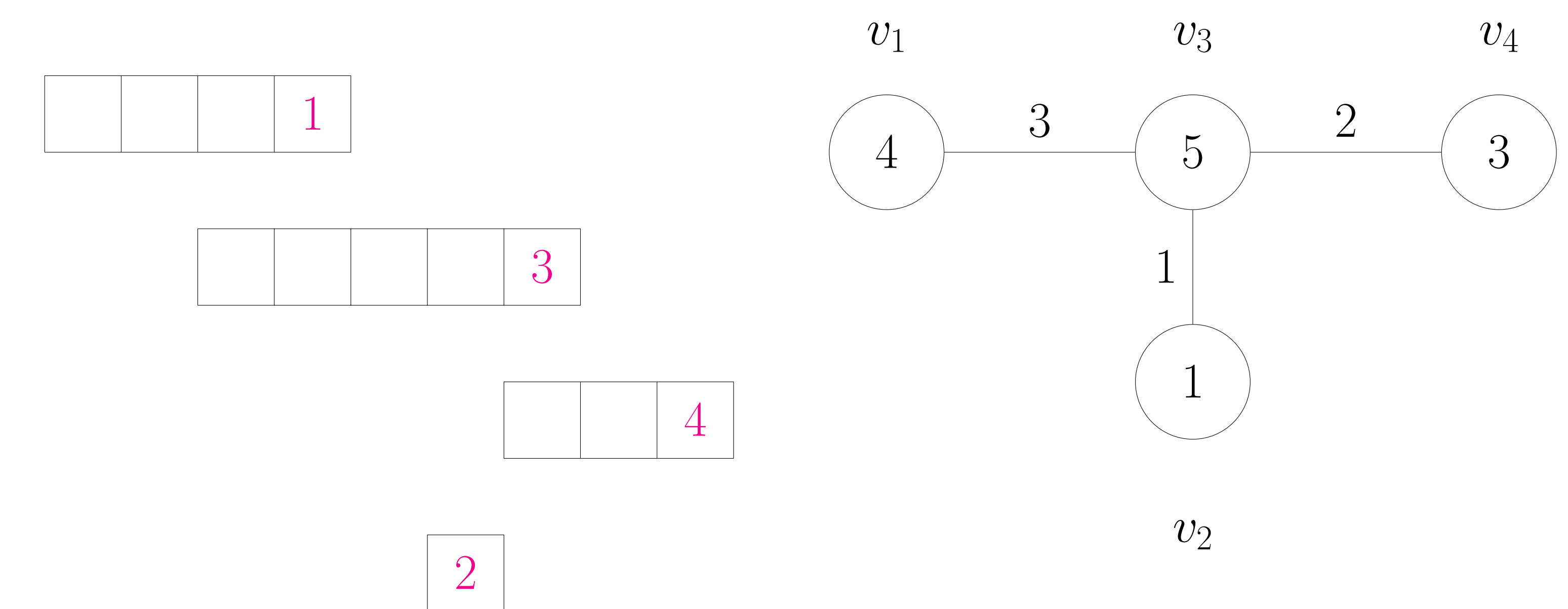
for a certain statistic  $\text{cocharge}_\Pi(T)$  on tableaux.

**Theorem: Plethystic relationship**

$$\text{We have } \left( \frac{G_\lambda([x(q-1)]; q)}{(q-1)^n} \right) \Big|_{q=1} = X_{\Pi(\lambda)}(x),$$

the extended chromatic symmetric function of the weighted graph  $\Pi(\lambda)$ .

## Example of combinatorial formula



Coefficient of  $s_{733}$ :  $(2q^5 + q^6)$

$$S = \begin{array}{cccc} 4 & 4 & 4 & \\ 3 & 3 & 3 & \\ 1 & 1 & 1 & 1 & 2 & 3 & 3 \end{array}$$

$$S|_{1,3} = \begin{array}{cccc} 3 & 3 & 3 & \\ 1 & 1 & 1 & 1 & 2 & 3 & 3 \end{array} \quad 2$$

$$S|_{2,3} = \begin{array}{cccc} 3 & & & \\ 2 & 3 & 3 & 3 & 3 \end{array} \quad 1$$

$$S|_{3,4} = \begin{array}{cccc} 4 & 4 & 4 & \\ 3 & 3 & 3 & 3 & 3 \end{array} \quad 2$$

$$\text{cocharge}_\Pi(S) = 2 + 1 + 2 = 5$$

$$T = \begin{array}{cccc} 4 & 4 & 4 & \\ 2 & 3 & 3 & \\ 1 & 1 & 1 & 1 & 3 & 3 & 3 \end{array}$$

$$T|_{1,3} = \begin{array}{cccc} 3 & 3 & 3 & \\ 1 & 1 & 1 & 1 & 3 & 3 & 3 \end{array} \quad 3$$

$$T|_{2,3} = \begin{array}{cccc} 2 & 3 & 3 & 3 & 3 \end{array} \quad 0$$

$$T|_{3,4} = \begin{array}{cccc} 4 & 4 & 4 & \\ 3 & 3 & 3 & 3 & 3 \end{array} \quad 2$$

$$\text{cocharge}_\Pi(T) = 3 + 0 + 2 = 5$$

$$U = \begin{array}{cccc} 3 & 4 & 4 & \\ 2 & 3 & 3 & \\ 1 & 1 & 1 & 1 & 3 & 3 & 4 \end{array}$$

$$U|_{1,3} = \begin{array}{cccc} 3 & & & \\ 2 & 3 & 3 & \\ 1 & 1 & 1 & 1 & 3 & 3 \end{array} \quad 3$$

$$U|_{2,3} = \begin{array}{cccc} 3 & & & \\ 2 & 3 & 3 & 3 & 3 \end{array} \quad 1$$

$$U|_{3,4} = \begin{array}{cccc} 4 & 4 & & \\ 3 & 3 & 3 & 3 & 3 & 4 \end{array} \quad 2$$

$$\text{cocharge}_\Pi(U) = 3 + 1 + 2 = 6$$

## References

- [1] I. Grojnowski and M. Haiman, *Affine Hecke algebras and positivity of LLT and Macdonald polynomials*. (2007).
  - [2] A. Lascoux, B. Leclerc, and J.-Y. Thibon, *Ribbon tableaux, Hall-Littlewood functions, quantum affine algebras, and unipotent varieties*. J. Math. Phys. 38 (1997), no. 2, 1041–1068.
  - [3] F. Tom, *A combinatorial Schur expansion of triangle-free horizontal-strip LLT polynomials*. Comb. Theory 1 14 arXiv:2011.13671 (2021).
  - [4] F. Tom, *A horizontal-strip LLT polynomial is determined by its weighted graph*. arXiv:2110.07984 (2021).
- E-mail: ffrom@mit.edu