Inequalities for f^*-vectors of Lattice Polytopes

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Definitions and Examples

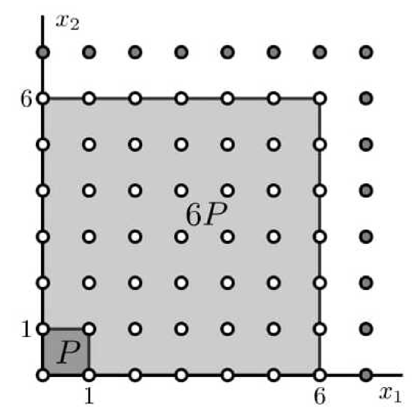
- $P \subset \mathbb{R}^d$ is a d-dimensional **lattice polytope**, i.e., the convex hull of finitely many points in $\mathbb{Z}^d$.
- $nP := \{np : p \in P\}$ is the n-th **dilate** of P, $n \in \mathbb{N}$.
- $\mathrm{ehr}_P(n) := |nP \cap \mathbb{Z}^d|$ is the **Ehrhart polynomial** of P.
- Expressed in different bases,

$$\mathrm{ehr}_P(n) = \sum_{k=0}^{d} h_k^* \binom{n+d-k}{d} = \sum_{k=0}^{d} f_k^* \binom{n-1}{k}.$$

- h^***-vector** of P: $h^*(P) = (h_0^*, h_1^*, \dots, h_d^*)$
- f^***-vector** of P: $f^*(P) = (f_{-1}^*, f_0^*, f_1^*, \dots, f_d^*)$, where $f_{-1}^* = 1$.
- Advantage: $h^*(P)$ and $f^*(P)$ are **non-negative** for all P. [1,2]
- $f_d^* = \mathrm{vol}(P) \cdot d!$ is the **normalized volume** of P
- The **degree** of P is the degree of $h^*(P;x) := \sum_{k=0}^{d} h_k^* x^k$.
- For a unimodular triangulation T of P:

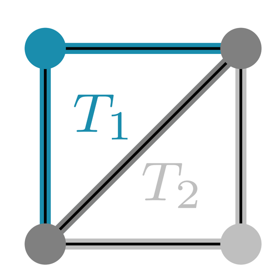
$$f_k^*(P) = f_k(T) := \#\{k\text{-dimensional faces in } T\}.$$

Example: The 6th dilate of $P = [0,1]^2$. [3]



- $\mathrm{ehr}_P(n) = (n+1)^2$
- $\begin{cases} (h_0^*, h_1^*, h_2^*) = (1,1,0) \\ (f_{-1}^*, f_0^*, f_1^*, f_2^*) = (1,4,5,2) \end{cases}$

Triangulation: $T = T_1 \cup T_2$



- $f(T) = (1,4,5,2)$

The Inequalities

Theorem 1

The f^*-vector of a d-dimensional lattice polytope satisfies

$$f_{-1}^* < f_0^* < f_1^* < \cdots < f_{\lfloor \frac{d}{2} \rfloor -1}^* \le f_{\lfloor \frac{d}{2} \rfloor}^* \quad \& \quad f_{\lfloor \frac{3d}{4} \rfloor}^* > \cdots > f_d^*$$

and

$$f_k^* \le f_{d-1-k}^*$$

for $0 \le k \le \frac{(d-3)}{2}$ and $d \ge 2$.

Moreover, if $h_d^* \neq 0$ and $h^*(P) \neq (1,1,...,1)$ then for $0 < k < \frac{d}{2}$,

$$f_k^* < f_{d-k}^* \quad \& \quad f_0^* \le f_d^*.$$

Note that

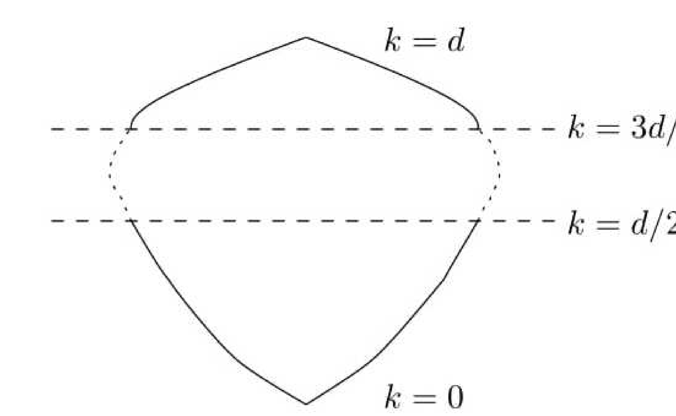
- $f_{\lfloor \frac{d}{2} \rfloor}^* > f_{\lfloor \frac{d}{2} \rfloor +1}^*$ holds among the coefficients of the f^*-vector of the d-dimensional simplex Δ.
- $f_{\lfloor \frac{3d}{4} \rfloor -1}^* < f_{\lfloor \frac{3d}{4} \rfloor}^*$ holds among the coefficients of the f^*-vector of the d-dimensional cube $P = [-1,1]^d$ for $d = 2$.

Similarities with f-vectors:

$f_k(P) = \#\{k\text{-dimensional faces in } P\}$.

- The f-vector of a **simplicial** (i.e., the faces are simplices) d-dimensional polytope satisfies all inequalities in Theorem 1 (Björner [5,6]).

In fact, the decrease starts from $\lfloor \frac{3(d-1)}{4} \rfloor - 1$.

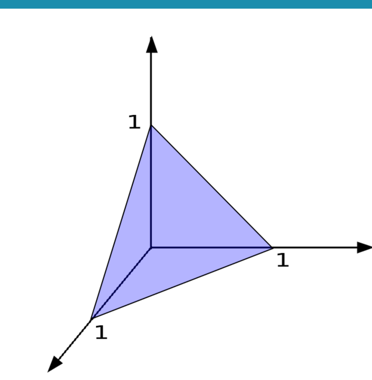


Poset visualisation [7]

About Symmetry

The standard simplex $\Delta := \mathrm{conv}\{e_1, ..., e_{d+1}\}$ (and its lattice equivalents) is the **only** lattice d-dimensional polytope with symmetric f^*-vector:

$$f^*(\Delta) = \left(1, \binom{d+1}{1}, \binom{d+1}{2}, \dots, \binom{d+1}{d+1}\right).$$



About Unimodality

The f^*-vector of P is called **unimodal** if $f_{-1}^* \le \cdots \le f_{p-1}^* \le f_p^* \ge f_{p+1}^* \ge \cdots \ge f_d^*$ for some p.

Unimodal examples

The f^*-vector of a d-dimensional polytope P of degree s is unimodal if:

- $s \le 5$ or
- $d \le 13$ or
- $d \ge 2s^2 - 2s - 2$.

Non-unimodal example

The f^*-vector of the simplex (introduced in [4]) $\Delta_w = \mathrm{conv}\{0, e_1, e_2, ..., e_{14}, w\}$, where

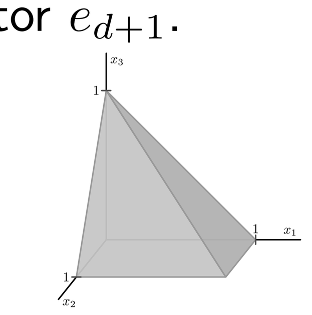
$$w = (\underbrace{1, \dots, 1}_{7}, \underbrace{131, \dots, 131}_{7}, 132),$$

satisfies $f_8^* > f_9^* < f_{10}^* > f_{11}^*$. Here $d = 15$.

Given a polytope $P \subset \mathbb{R}^d$, we denote by $\mathrm{Pyr}(P) \subset \mathbb{R}^{d+1}$ the convex hull of P and the unit vector e_{d+1}.

Proposition

For any lattice polytope P, $\mathrm{Pyr}^n(P)$ has unimodal f^*-vector for sufficiently large n.



Gorenstein Polytopes

P is **Gorenstein** of index g, $g \ge 1$, whenever $h^*(P;x)$ has degree $d+1-g$ and is symmetric with respect to its degree.

Example: $h^*([0,1]^2; x) = 1 + x$ hence $[0,1]^2$ is Gorenstein of index 2.

Theorem 2

The f^*-vector of a d-dimensional lattice polytope that is Gorenstein of index g satisfies

$$f_k^* > \cdots > f_{\lfloor \frac{3d-1}{4} \rfloor}^* > \cdots > f_d^*, \qquad \text{for } k = \frac{1}{2}\left(d - 1 + \left\lfloor \frac{d+1-g}{2} \right\rfloor\right).$$

Future Work

- Compute more *examples* of f^*-vectors and look for a combinatorial interpretation.
- Is $f^*(P)$ unimodal when P admits a *unimodular triangulation*?
- Are there polytopes with unimodal h^*-vector and *nonunimodal* f^*-vector? (The converse is not true.)
- Is there a polytope of *dimension* 14 with non-unimodal f^*-vector?
- What about *log-concavity* or *real-rootedness* of the f^*-polynomial of $\mathrm{Pyr}^n(P)$ for sufficiently large n?
- Can we understand the f^*-vector of the *interior* or the *boundary* of a polytope?

References

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