An Algebraic Perspective on Ramsey Numbers

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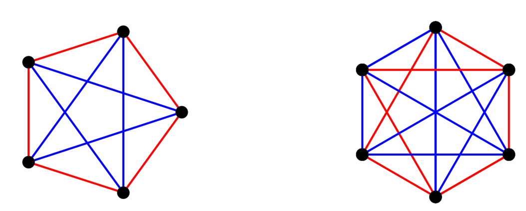
Introduction

- For positive integers r and s, the classical **Ramsey number** $R(r,s)$ is the smallest n such that every edge 2-coloring of the edges of K_n contains either a clique of size r in the first color or a clique of size s in the second color.

Example. $R(3,3) = 6$:



- Few values of $R(r,s)$ are known:

r \ s	1	2	3	4	5	6	7	8	9	
1	1	1	1	1	1	1	1	1	1	...
2		2	3	4	5	6	7	8	9	...
3			6	9	14	18	23	28	36	
4				18	25					

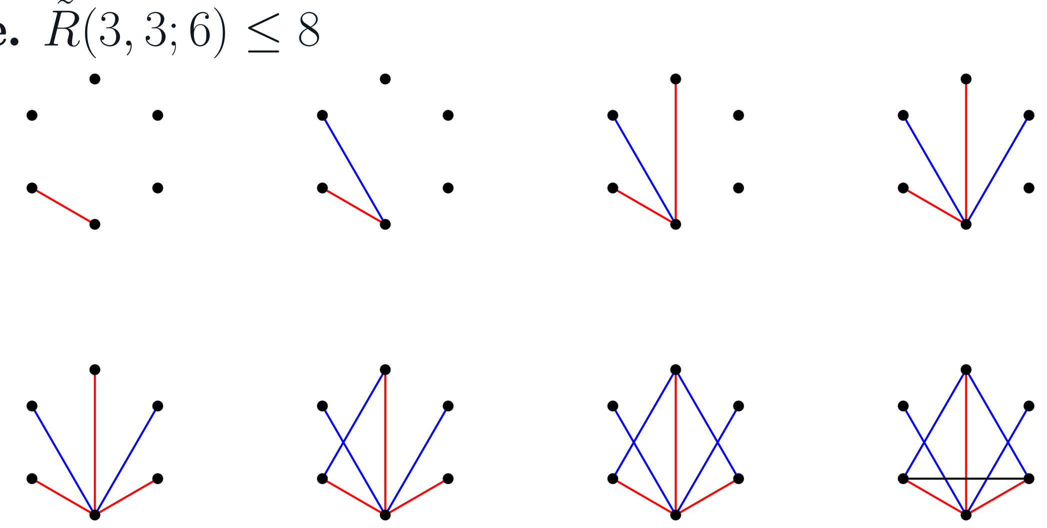
- **Goal:** use methods from algebraic geometry to study how difficult it is to **certify** Ramsey number bounds

Restricted Online Ramsey Numbers

Game between the two players **Builder** and **Painter**:

- Fix positive integers r and s.
- Builder and Painter take turns where Builder selects an edge of K_n and Painter colors it either red or blue.
- Builder wins when there is a red K_r or blue K_s.
- The **restricted online Ramsey number** $\tilde{R}(r,s;n)$ is the smallest number of turns for which Builder is guaranteed a victory.
- Builder always wins the game eventually if $n \geq R(r,s)$.

Example. $\tilde{R}(3,3;6) \leq 8$



Polynomial Encoding

System of **polynomial equations** that has solution if and only if $R(r,s) > n$.

- Variables x_e, y_e, $e \in E(K_n)$ take on 0/1 values
- x_e set to 1 if and only if e colored red
- y_e set to 1 if and only if e colored blue
- We have $R(r,s) \leq n$ if and only if there is no solution to the following system over $\overline{\mathbb{F}}_2$, where $K_n = (V,E)$ is the complete graph on n vertices:

$$\prod_{e \in E(S)} x_e = 0 \qquad \forall S \subseteq V, |S| = r$$
$$\prod_{e \in E(S)} y_e = 0 \qquad \forall S \subseteq V, |S| = s$$
$$1 + x_e + y_e = 0 \qquad \forall e \in E$$

Nullstellensatz Certificates

Theorem: Hilbert, 1893

Let K be an algebraically closed field, and let $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$. Then there is no solution to the system $f_1 = \cdots = f_m = 0$ if and only if there exist polynomials $\beta_1, \ldots, \beta_m$ such that $\sum_{i=1}^m \beta_i f_i = 1$.

- The identity $\sum_{i=1}^m \beta_i f_i = 1$ is called a *Nullstellensatz certificate.* The *degree* of the certificate is the maximum degree of the β_i.
- Nullstellensatz certificate degrees for combinatorial problems such as 3-coloring are often small "in practice" [1]

Main Result

Theorem: De Loera-Wesley

Using the previous encoding, there exists a Nullstellensatz certificate of degree at most $\tilde{R}(r,s;n) - 1$ for $n = R(r,s)$.

The restricted online Ramsey numbers are known to be strictly smaller than the number of edges in $K_{R(r,r)}$ [2]:

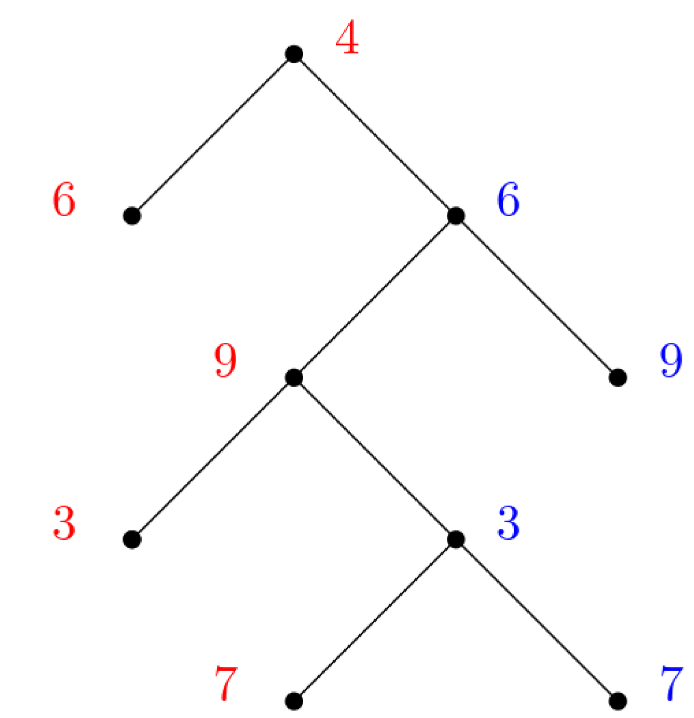
$$\tilde{R}(r,r;n) \leq \binom{n}{2} - \Omega(n \log n) \text{ when } n = R(r,r).$$

Additional Results

Similar results hold for other Ramsey-type numbers:

- **Multicolor Ramsey numbers:** $R(r_1, \ldots, r_k)$ = smallest n such that every edge k-coloring of K_n contains a monochromatic K_{r_i} in color i for some i.
- **Rado numbers:** $R_k(\mathcal{E})$ = smallest n such that every k-coloring of $\{1, \ldots, n\}$ contains a monochromatic solution to an equation $\mathcal{E}$.
- **van der Waerden numbers:** $w(k,\ell)$ = smallest n such that every k-coloring of $\{1, \ldots, n\}$ contains a monochromatic arithmetic progression of length ℓ.
- We define a Builder-Painter game and analogues of the restricted online Ramsey numbers for these other Ramsey-type numbers.

Example. Let $\mathcal{E}$ be the equation $x+3y = 3z$. The tree below describes an optimal strategy for Builder for two colors:



- The minimal degree of a Nullstellensatz certificate for the corresponding encoding is therefore at most 4. Computations show the minimal degree is in fact 2.
- The following inequalities are strict in general:

minimal Nullstellensatz degree $\leq$ restricted online Rado number
$\leq$ Rado number

References and Acknowledgements

[1] J. A. De Loera et al. "Expressing Combinatorial Problems by Systems of Polynomial Equations and Hilbert's Nullstellensatz". In: *Combinatorics, Probability and Computing* 18.4 (2009), pp. 551–582.

[2] D. Gonzalez, X. He, and H. Zheng. "An upper bound for the restricted online Ramsey number". In: *Discrete Math.* 342.9 (2019), pp. 2564–2569.

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