CM regularity for 321-avoiding Kazhdan–Lusztig varieties

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Introduction

We introduce an algorithm to compute the degrees of 321-avoiding unspecialized Grothendieck polynomials. Our result provides an algorithm to compute the CM regularity of 321-avoiding Kazhdan–Lusztig ideals. This extends the work of an earlier paper of Rajchgot, the author, and Weigandt (2023) which gives a formula in the Grassmannian case.

Kazhdan–Lusztig varieties

Consider the Schubert variety X_u and the opposite Schubert cell Ω_w° in $\mathbf{GL}_n(\mathbb{C})$.

Theorem 1 (Kazhdan-Lusztig, '79)

$$X_u \cap w\Omega_{id}^\circ \cong (X_u \cap \Omega_w^\circ) \times \mathbb{C}^{\ell(w)}$$

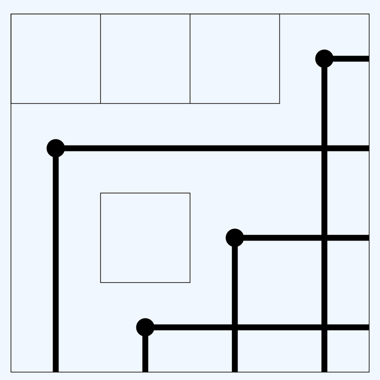
Of particular interest is the **Kazhdan-Lusztig variety**

$$\mathcal{N}_{u,w} = X_u \cap \Omega_w^\circ$$

defined by Woo-Yong '06. For $w \leq u$, $\mathcal{N}_{u,w}$ has defining ideal

$$I_{u,w} = \langle r_w(i,j) + 1 \text{ minors of } \mathrm{z}_{i\times j}(u)\rangle.$$

Example 2 *Consider* $w = 4132, u = 4231.$



$$\xrightarrow{r_w} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix} \xrightarrow{z(u)} \begin{pmatrix} z_{11} & z_{12} & z_{13} & 1 \\ z_{21} & 1 & 0 & 0 \\ z_{31} & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$I_{u,w} = \langle z_{11},\ z_{12}, z_{13},\ z_{11} - z_{12}z_{21},\ -z_{12}z_{31},\ -z_{31}\rangle$$

For coordinate ring S/I, where S is standard graded, consider its **minimal free resolution**

$$0 \to \bigoplus_j S(-j)^{\beta_{l,j}} \to \cdots \to \bigoplus_j S(-j)^{\beta_{0,j}} \to S/I \to 0.$$

The algebraic invariant **CM regularity** of S/I is given by

$$\mathrm{reg}(S/I) := \max\{j - i \mid \beta_{i,j} \neq 0\}.$$

Using Benedetti–Varbaro '15 and Woo–Yong'12:

Proposition 3 (Rajchgot-R.-Weigandt '23) *For* $u, w \in S_n$ *321-avoiding*

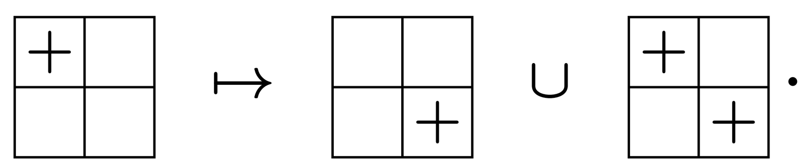
$$\mathrm{reg}(\mathbb{C}[\mathcal{N}_{u,w}]) = \deg(\mathfrak{G}_{u,w}) - \ell(w),$$

where $\mathfrak{G}_{u,w}$ *is the unspecialized Grothendieck polynomial.*

Connection to Excited Young Diagrams

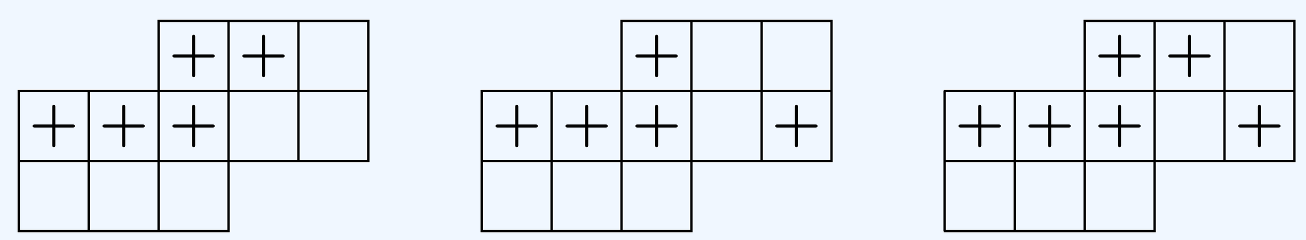
Let $u \geq w \in S_n$ be 321-avoiding. Define $\mathcal{R}_u$ as the skew Young diagram associated to u. Place $+$'s in $\mathcal{R}_u$ marking the earliest subword of w in u. Call this diagram $D_{\mathrm{top}}(u,w)$.

A **K-excited move** on a diagram D in $\mathcal{R}_u$ is the local operation on a 2×2 subsquare of D such that



A **K-skew excited Young diagram** of w in u is a diagram obtainable via successive K-excited moves on $D_{\mathrm{top}}(u,w)$.

Example 4 *Below are the diagrams corresponding to* $u = 47128356$ *and* $w = 14273568$.



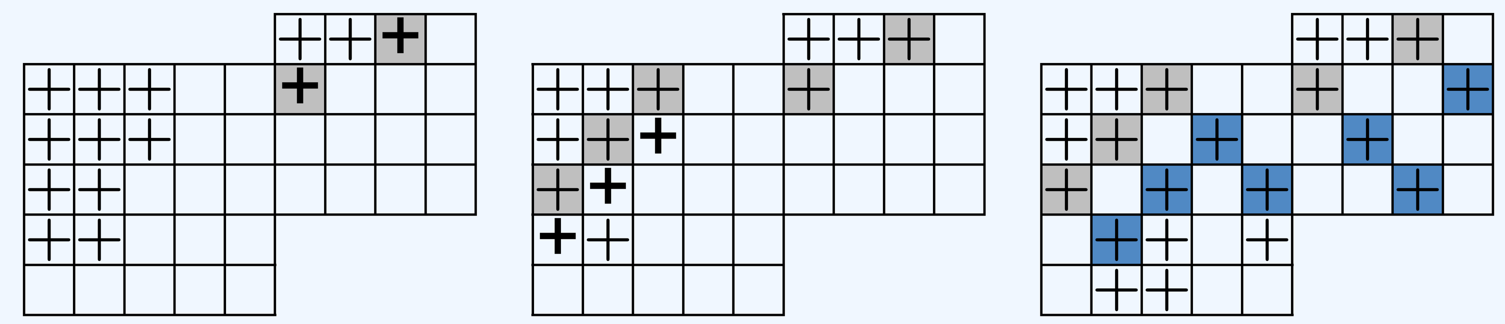
Call the collection of these diagrams for u, w $\overline{\mathsf{SEYD}}(u,w)$.

Proposition 5 (R., '23) *For* $u \geq w \in S_n$ *321-avoiding*

$$\deg(\mathfrak{G}_{u,w}) = \max\{\#D \mid D \in \overline{\mathsf{SEYD}}(u,w)\}.$$

We give an algorithm to construct $D \in \overline{\mathsf{SEYD}}(u,w)$ to maximize $\#D$, the number of $+$'s in D. This algorithm computes a statistic $\Delta_{u,w}$ in terms of maximal antidiagonals in D.

Example 6 *Below we illustrate our algorithm for particular* $u, w \in S_{15}$,



Then $\deg(\mathfrak{G}_{u,w}) = \ell(w) + (3+4) = 14 + 7 = 21.$

Combining Propositions 3 and 5 with our algorithm:

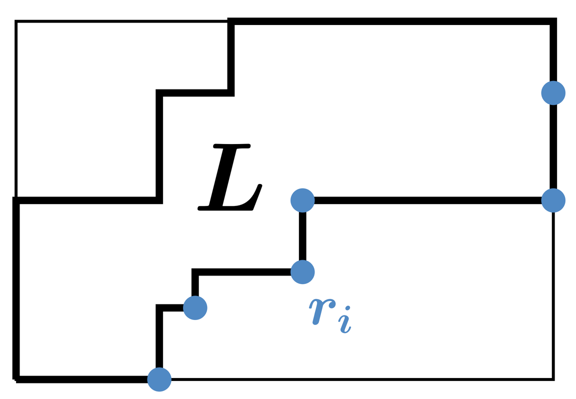
Theorem 7 (R., '23) *Let* $u \geq w \in S_n$ *be 321-avoiding. Suppose* $D_{\mathrm{top}}(u,w)$ *has* s *connected components. Then*

$$\mathrm{reg}(\mathbb{C}[\mathcal{N}_{u,w}]) = \sum_{q\in[s]} |\Delta_{u,w}(q)|.$$

Thus for u, w as in Example 6, $\mathrm{reg}(\mathbb{C}[\mathcal{N}_{u,w}]) = 7$.

Application

A two-sided ladder L is a skew-Young diagram filled with indeterminates. The ideal I_L is generated by the northwest r_i minors of L and defines the **two-sided mixed ladder determinantal variety** $\mathbb{C}[L]/I_L$.



Escobar–Fink–Rajchgot–Woo '23+ prove that $\mathbb{C}[L]/I_L \cong \mathbb{C}[\mathcal{N}_{u(L),w(L)}]$ where $u(L), w(L) \in S_n$ are 321-avoiding.

Corollary 8 (R., '23) *For a two-sided ladder* L,

$$\mathrm{reg}(\mathbb{C}[L]/I_L) = \sum_{q\in[s]} |\Delta_{u(L),w(L)}(q)|.$$

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