Motivation

The irreducible representations of the symmetric group $S_n$ are the Specht modules $S^n$ indexed by integer partitions $\lambda \vdash n$. For the case of 3-row rectangles, Kuperberg [5] famously introduced a diagrammatic "web" basis of the Specht module $S^3$ (and more generally for other spaces of invariant tensors). Kuperberg's web basis has many important applications to quantum link invariants, cluster algebras, and algebraic geometry. From a combinatorial perspective, a key property of the web basis are that the long cycle $c = (1, 2 \ldots n)$ acts diagrammatically as a rotation $[6]$. Our main result is a rotation-invariant web basis for the 4-row rectangular Specht module $S^{4,3}$ (and also for more general spaces of tensor invariants) $[6]$.

Hourglass Plabic Graphs

An hourglass plabic graph is a bipartite planar graph embedded in a disc with black boundary vertices, with hourglass edges $\leftrightarrow$ allowed between internal vertices, and whose internal vertices are all 4-valent. Hourglass plabic graphs admit moves of two kinds: benzene moves and square moves.

Two hourglass plabic graphs $G$ and $G'$ are equivalent, $G \sim G'$, if one may be obtained from the other by a sequence of square and benzene moves. An hourglass plabic graph is top if every benzene face is oriented as in the figure above on the top left. An hourglass plabic graph $G$ is contracted fully reduced (CRG) if no $G' \sim G$ contains any of the following substructures:

1. An interior vertex incident to fewer than three other vertices,
2. A 2-cycle (treating an hourglass edge as a single edge), or
3. A 4-cycle containing an hourglass edge.

Trip Permutations

A key feature of hourglass plabic graphs is that they have three trip permutations $trip_i(G)$, $trip_j(G)$, and $trip_k(G)$ defined using the rules of the road. For $trip_i(G)$:

1. Start at vertex $\lambda_i$.
2. Follow the edges in $G$ and turn at each internal right.
3. Take the $i$-th left at white vertices and $i$-th right at black vertices.
4. The process ends at boundary vertex $\lambda_i$. Then set $trip_i(G)(x) = x$.

Promotion Permutations

Promotion permutations keep track of the entries in $E'$ sliding from row $i + 1$ to row $i$ when applying Schützenberger promotion [3].

For an equivalence class of CRG, the set of promotion permutations for $G(E) = \{E_0, E_1, \ldots, E_n\}$ is the set of bijections:

$$\forall i \in \{0, 1, \ldots, n\}, \text{promotion}(E_i) = \text{promotion}(E_{i+1})$$

$E$ =

$$\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{bmatrix}$$

$P(E)$ =

$$\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{bmatrix}$$

Tensor Invariants

To each hourglass plabic graph $G$ with $n$ boundary vertices, associate a polynomial $[G]$ in

$$\text{Inv}((C_s)^{\otimes n}) = \text{Hom}_{S_n}(C_s, C)^{\otimes n} \subset \mathbb{C}[x_1, x_2, x_3, x_4; 1 \leq i \leq n]$$

which is invariant under the action of $S_n$.

Recall that as an $S_n$-module: $\text{Inv}((C_s)^{\otimes n}) \cong S^{n+k(n)}$.

Main Theorems

Theorem: $T$ and $G$ are mutually inverse bijections between move-equivalence classes of contracted fully reduced hourglass plabic graphs and 4-row rectangular SYT.

Furthermore, this bijection satisfies $\text{inv}(G) = \text{inv}(T(G))$ and consequently intertwines promotion of tableaux with rotation of hourglass plabic graphs.

Theorem: The invariant polynomials $[G]^{\text{inv}}_s$ of top contracted fully reduced hourglass plabic graphs with $n$ boundary vertices are a rotation-invariant web basis for the invariant space $\text{Inv}((C_s)^{\otimes n})$.

Symmetrized Six-Vertex Configurations and ASMs

A CRG is transformed into its symmetrized six-vertex configurations by orienting edges from its black vertex to its white vertex and contracting hourglasses. The elements in the move-equivalence class of the SYT filled with 1, 2, 3, 4 from left to right and top to bottom are in bijection with alternating sign matrices.

Benzene Equivalence Class and Plane Partitions

Hexagonal regions in hourglass plabic graphs can be identified with plane partitions. The benzene move corresponds to adding/removing boxes from the plane partition.

References