# An $\mathrm{SL}_4$-web basis from hourglass plabic graphs

Christian Gaetz [1] Oliver Pechenik [2] Stephan Pfannerer [3] Jessica Striker [4] Joshua P. Swanson [5]

[1]Cornell University [2]University of Waterloo [3]TU Wien [4]North Dakota State University [5]University of Southern California

## Motivation

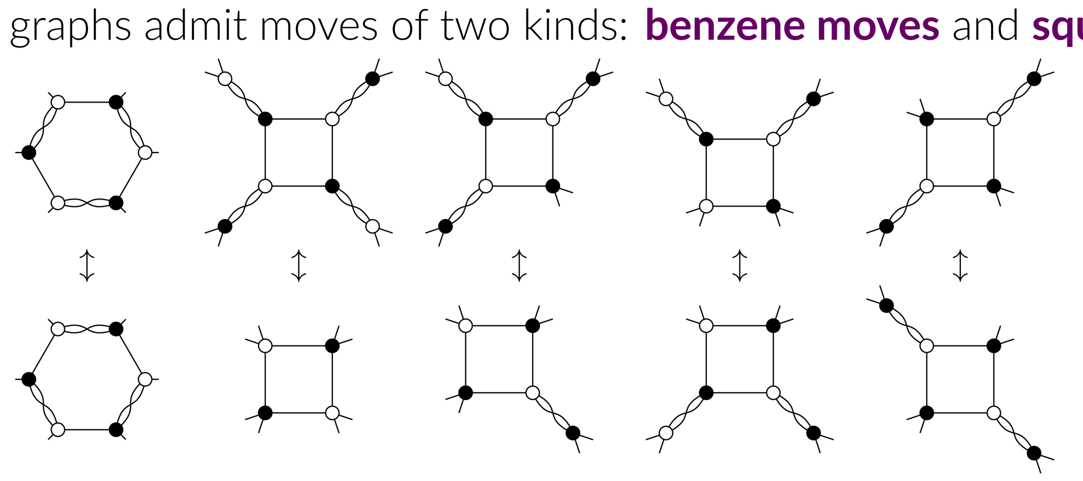
The irreducible representations of the symmetric group $S_n$ are the *Specht modules* $S^\lambda$ indexed by integer partitions $\lambda \vdash n$. For the case of 3-row rectangles, Kuperberg [5] famously introduced a diagrammatic "web" basis of the Specht module $S^{3\times b}$ (and more generally for other spaces of invariant tensors). Kuperberg's web basis has many important applications to quantum link invariants, cluster algebras, and algebraic geometry. From a combinatorial perspective, a key property of the web basis are that the long cycle $c = (12\ldots n)$ acts diagrammatically as a rotation [6].

Our main result is a rotation-invariant web basis for the 4-row rectangular Specht module $S^{4\times b}$ (and also for more general spaces of tensor invariants) [4].

## Hourglass Plabic Graphs

An **hourglass plabic graph** is a bipartite planar graph embedded in a disc with black boundary vertices, with hourglass edges allowed between internal vertices, and whose internal vertices are all 4-valent.

Hourglass plabic graphs admit moves of two kinds: **benzene moves** and **square moves**.

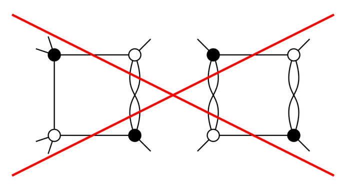


Two hourglass plabic graphs $G$ and $G'$ are **equivalent**, $G \sim G'$, if one may be obtained from the other by a sequence of square and benzene moves.

An hourglass plabic graph is **top** if every benzene face is oriented as in the figure above on the top left.

An hourglass plabic graph $G$ is **contracted fully reduced (CRG)** if no $G' \sim G$ contains any of the following substructures:

1. An interior vertex incident to fewer than three other vertices,
2. A 2-cycle (treating an hourglass edge as a single edge), or
3. A 4-cycle containing an hourglass edge.



## Trip Permutations

A key feature of hourglass plabic graphs is that they have *three* **trip permutations** $\mathrm{trip}_1(G), \mathrm{trip}_2(G)$, and $\mathrm{trip}_3(G)$ defined using the rules of the road. For $\mathrm{trip}_i(G)$:

1. Start at vertex $b_s$,
2. Follow the edges in $G$ and turn at each internal vertex,
3. Take the $i$-th left at white vertices and $i$-th right at black vertices.
4. The process ends at boundary vertex $b_t$. Then set $\mathrm{trip}_i(G)(s) = t$.

## Schützenberger Promotion

$E =$

| | | | |
|---|---|---|---|
| 1 | 2 | 6 | 8 |
| 3 | 5 | 7 | 14 |
| 4 | 9 | 10 | 15 |
| 11 | 12 | 13 | 16 |

$\to$

| | | | |
|---|---|---|---|
| | 2 | 6 | 8 |
| 3 | 5 | 7 | 14 |
| 4 | 9 | 10 | 15 |
| 11 | 12 | 13 | 16 |

$\to$

| | | | |
|---|---|---|---|
| 2 | | 6 | 8 |
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$\to$

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|---|---|---|---|
| 2 | 5 | 6 | 8 |
| 3 | | 7 | 14 |
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| 2 | 5 | 6 | 8 |
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| 2 | 5 | 6 | 8 |
| 3 | 7 | 10 | 14 |
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| 2 | 5 | 6 | 8 |
| 3 | 7 | 10 | 14 |
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$\leftarrow$

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|---|---|---|---|
| 2 | 5 | 6 | 8 |
| 3 | 7 | 10 | 14 |
| 4 | 9 | 13 | 15 |
| 11 | 12 | 16 | |

$\leftarrow$

| | | | |
|---|---|---|---|
| 2 | 5 | 6 | 8 |
| 3 | 7 | 10 | 14 |
| 4 | 9 | 13 | 15 |
| 11 | 12 | 16 | 17 |

$\leftarrow$

$\mathcal{P}(E) =$

| | | | |
|---|---|---|---|
| 1 | 4 | 5 | 7 |
| 2 | 6 | 9 | 13 |
| 3 | 8 | 12 | 14 |
| 10 | 11 | 15 | 16 |

## Promotion Permutations

**Promotion permutations** keep track of the entries in $E$ sliding from row $i+1$ to row $i$ when applying *Schützenberger promotion* [3].

$$\mathrm{prom}_1(E) = 5\ 3\ 4\ 14\ 13\ 7\ 10\ 9\ 12\ 11\ \underline{6}\ \underline{2}\ \underline{1}\ 15\ 16\ \underline{8},$$
$$\mathrm{prom}_2(E) = 10\ 4\ 13\ \underline{2}\ 15\ 9\ 12\ 11\ \underline{6}\ \underline{1}\ \underline{8}\ \underline{7}\ \underline{3}\ 16\ \underline{5}\ \underline{14}, \text{ and}$$
$$\mathrm{prom}_3(E) = 13\ 12\ \underline{2}\ \underline{3}\ \underline{1}\ 11\ \underline{6}\ 16\ \underline{8}\ \underline{7}\ \underline{10}\ \underline{9}\ \underline{5}\ \underline{4}\ \underline{14}\ \underline{15}.$$

The *antiexcedances* $\mathrm{Aexc}$ of each permutation are underlined.

## Tensor Invariants

To each hourglass plabic graph $G$ with $n$ boundary vertices, associate a polynomial $[\![G]\!]$ in

$$\mathrm{Inv}((\mathbb{C}^4)^{\otimes n}) = \mathrm{Hom}_{\mathrm{SL}_4(\mathbb{C})}((\mathbb{C}^4)^{\otimes n}, \mathbb{C}) \subset \mathbb{C}[x_{i1}, x_{i2}, x_{i3}, x_{i4} : 1 \le i \le n]$$

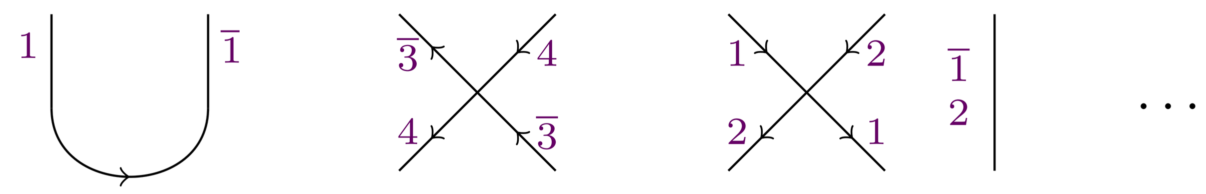
which is invariant under the action of $\mathrm{SL}_4$.

Recall that as an $S_n$-module: $\mathrm{Inv}((\mathbb{C}^4)^{\otimes n}) \cong S^{(4\times(n/4))}$.

## Bijections

$\mathcal{T}$ : **CRG/$\sim$ $\to$ SYT:** For an equivalence class $[G]$ of contracted fully reduced hourglass plabic graphs with $4b$ boundary vertices, there is a unique $4\times b$ standard Young tableau $\mathcal{T}([G])$ whose first $i$ rows contain the entries $\mathrm{Aexc}(\mathrm{trip}_i(G))$ for $i = 1, 2, 3$.

$\mathcal{G}$ : **SYT $\to$ CRG/$\sim$:** Given a rectangular SYT $T$ with four rows, we first translate it into a row of dangling strands labelled with the lattice word corresponding to $T$. Growth rules are applied to combine dangling strands until no further rules can be applied giving $\mathcal{G}(T)$.

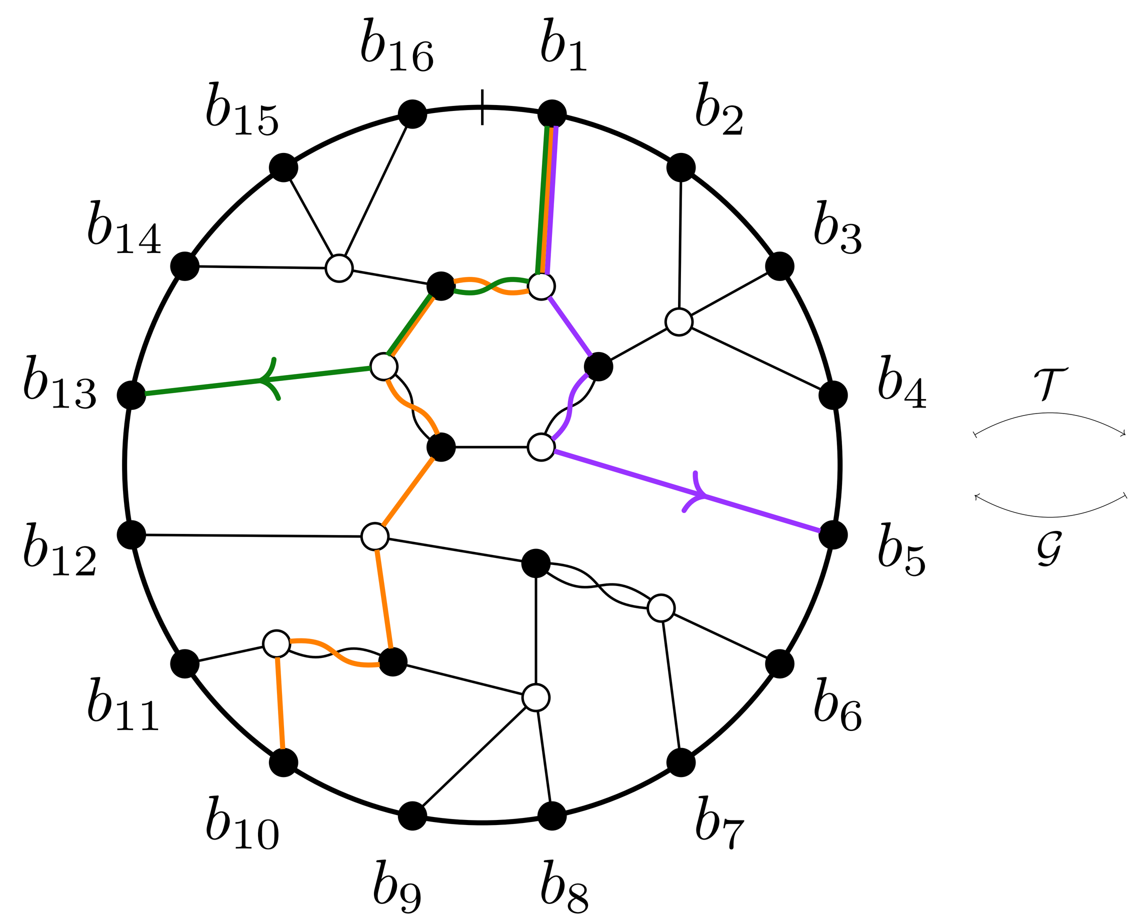



## Main Theorems

**Theorem:** $\mathcal{T}$ and $\mathcal{G}$ are mutually inverse bijections between move-equivalence classes of contracted fully reduced hourglass plabic graphs and 4-row rectangular SYT.

Furthermore, this bijection satisfies $\mathrm{trip}_\bullet(G) = \mathrm{prom}_\bullet(\mathcal{T}(G))$ and consequently intertwines promotion of tableaux with rotation of hourglass plabic graphs.

**Theorem:** The invariant polynomials $[\![G^{\mathrm{top}}]\!]$ of top contracted fully reduced hourglass plabic graphs with $n$ boundary vertices are a **rotation-invariant web basis** for the invariant space $\mathrm{Inv}((\mathbb{C}^4)^{\otimes n})$.

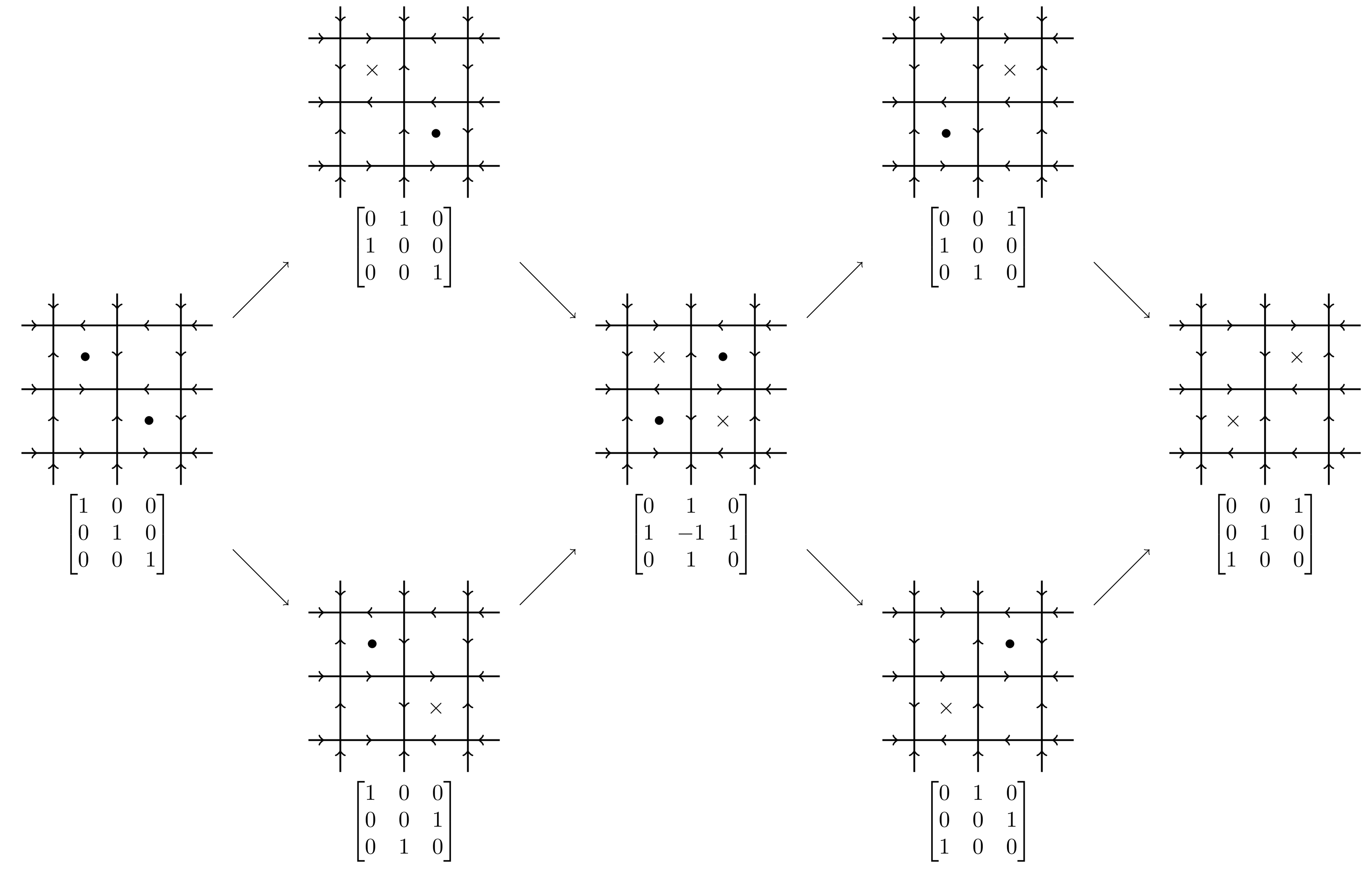



| | | | |
|---|---|---|---|
| 1 | 2 | 6 | 8 |
| 3 | 5 | 7 | 14 |
| 4 | 9 | 10 | 15 |
| 11 | 12 | 13 | 16 |

## Symmetrized Six-Vertex Configurations and ASMs

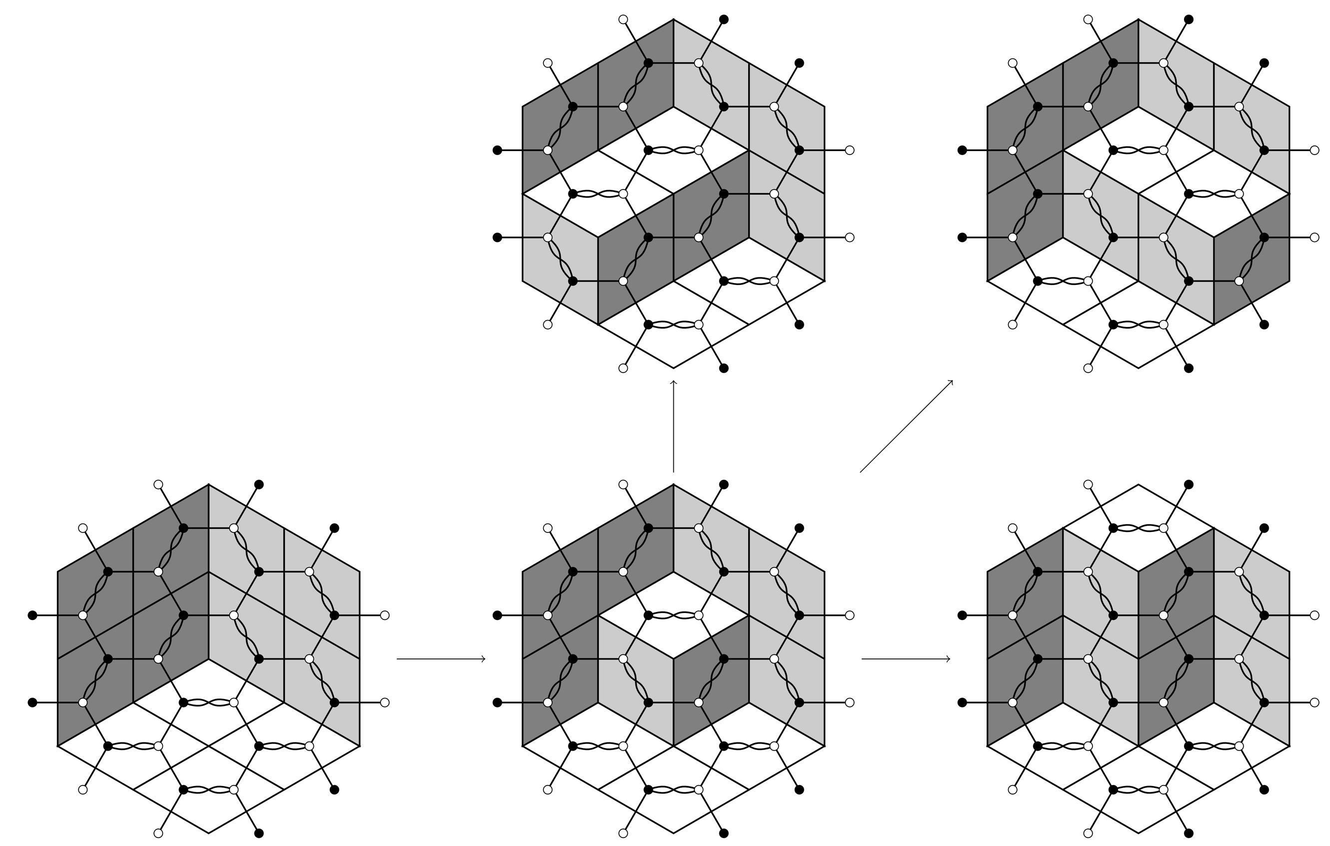
A CRG is transformed into its **symmetrized six-vertex configurations** by orienting edges from its black vertex to its white vertex and contracting hourglasses.

The elements in the move-equivalence class of the SYT filled with $1, 2, \ldots, n$ from left to right and top to bottom are in bijection with *alternating sign matrices*.



## Benzene Equivalence Class and Plane Partitions

Hexagonal regions in hourglass plabic graphs can be identified with plane partitions. The benzene move corresponds to adding/removing boxes from the *plane partition*.



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