Volume rigidity and algebraic shifting

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Volume computation

Let

$$\mathbf{p}: V(K) \to \mathbb{R}^{d-1}$$

be a mapping of the vertices of K.
The (signed) **volume** of $\sigma = \{v_1, \dots, v_d\} \in K$ with respect to $\mathbf{p}$ is given by the determinant of the $d \times d$ matrix

$$M_{\mathbf{p},\sigma} = \begin{pmatrix} \mathbf{p}(v_1) & \dots & \mathbf{p}(v_d) \\ 1 & \dots & 1 \end{pmatrix}.$$

Question

Is there a **non-trivial** continuous motion of the vertices starting at $\mathbf{p}$ that preserves the volumes of all the $(d-1)$-simplices in K? By "non-trivial" we mean that the volume of some $(d-1)$ non-face would change.

Volume-rigidity matrix

The **volume-rigidity matrix** $\mathfrak{V}(K, \mathbf{p})$ of the pair $(K, \mathbf{p})$ is a $(d-1)n \times f_{d-1}(K)$ matrix given by the Jacobian of the function $\mathbf{p} \mapsto (\det M_{\mathbf{p},\sigma})_{\sigma \in K}$, viewing $\mathbf{p}$ as a $(d-1)n$-dimensional vector. The column vector $\mathbf{v}_\sigma$ corresponding to a $(d-1)$-face $\sigma = \{v_1, \dots, v_d\} \in K$ is defined by

$$(\mathbf{v}_\sigma)_{v,j} = \begin{cases} C_{i,j}(M_{\mathbf{p},\sigma}) & \text{if } v = v_i \text{ and } j \in [d-1], \\ 0 & \text{otherwise.} \end{cases}$$

An n-vertex $(d-1)$-dimensional simplicial complex K is called **volume-rigid** if

$$\operatorname{rank}(\mathfrak{V}(K, \mathbf{p})) = (d-1)n - (d^2 - d - 1),$$

for a generic $\mathbf{p}: V(K) \to \mathbb{R}^{d-1}$.

Question

Can we characterize volume-rigidity in terms of algebraic shifting?

Example: Appolonian network

An Appolonian network is a 2-dimensional simplicial complex obtained by iteratively subdividing a triangle into three via a new vertex.

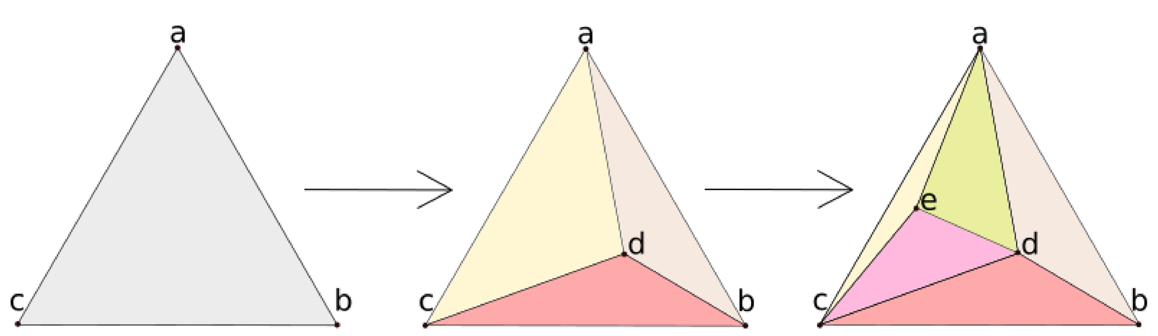

Figure 1:Example of Appolonian network. First we add vertex d creating triangles abd, acd, bcd and removing the triangle abc. Next we add vertex e creating triangles ade, ace, cde and removing triangle acd.

In the figure above, in the final complex the quantities vol(abc) and vol(acd) are preserved under continuous motions preserving all 2-faces. Is vol(bce) also preserved under such motions? Yes, since Appolonian networks are volume-rigid [2].

Exterior algebraic shifting

Consider the **exterior face ring**

$$\bigwedge K = \bigwedge \mathbb{R}^n / \langle e_S : S \notin K \rangle.$$

and let $q: \wedge \mathbb{R}^n \to \wedge K$ denote the natural quotient map. Let $<_p$ denote the partial order on $2^{[n]}$ defined as follows: $\sigma = \{\sigma_1 < \dots < \sigma_m\}, \tau = \{\tau_1 < \dots < \tau_m\} \in 2^{[n]}$, $\sigma <_p \tau$ if $\sigma_i \le \tau_i$ for all $i \in [m]$. Let $(f_i)_{i \in [n]}$ be a generic change of basis in $\mathbb{R}^n$.
The **exterior algebraic shifting of** K **w.r.t.** $<_p$ is defined by

$$\Delta^{<_p} K = \{\sigma \subseteq [n] : q(f_\sigma) \notin \operatorname{span}_{\mathbb{R}}\{q(f_\tau) : \tau <_p \sigma\}\}.$$

Shifting w.r.t. a partial order has several of the good properties of shifting with respect to a linear order because

$$\Delta^{<_p} K = \bigcup_{<_l \in \mathcal{L}} \Delta^{<_l} K,$$

where $\mathcal{L}$ is the set of all linear extension of $<_p$.

Theorem (B., Nevo and Peled [1])

Fix $d \ge 3$. An n-vertex $(d-1)$-dimensional simplicial complex K is volume-rigid if and only if $\{1, 3, 4, \dots, d, n\} \in \Delta^{<_p}(K)$.

Proof method

We identify the generic placement of the vertices $\mathbf{p}$ with the generic change of basis $(f_i)_{i \in [n]}$. First, we set $f_1 = \mathbf{1}$. Next, for each $v \in V(K)$ we set the entry of f_{i+1} corresponding to v equal to the i-th entry of $p(v)$, i.e. $(f_{i+1})_v = p(v)_i$. Define

$$\psi: \bigoplus_{i=2}^{d} \bigwedge^{1} \mathbb{R}^n \to \bigwedge^{d} \mathbb{R}^n$$

$$(m_2, \dots, m_d) \mapsto \sum_{i=2}^{d} f_{[d] \setminus \{i\}} \wedge m_i.$$

Since

$$\langle e_\sigma, \psi(e_{i,v}) \rangle = (-1)^{i-1} C_{i-1,j}(M_{\mathbf{p},\sigma})$$

we have that the matrix representation of $q \circ \psi$ is equal to the transpose of the volume-rigidity matrix $\mathfrak{V}(K, \mathbf{p})$.

Local moves

Edge contraction. Let K be a pure $(d-1)$-dimensional simplicial complex, $e = \{u, w\} \in K$ such that at least $(d-1)$ facets in K contain e. Let K' be the simplicial complex obtained from K by contracting the edge e, i.e. by identifying the vertex u with w, and removing duplicates. If K' is volume rigid then so is K.
Vertex addition. Let K be $(d-1)$-volume-rigid, $v \notin V(K)$ and $S \subseteq V(K)$ such that $|S| \ge d$, then $K \cup (v * \binom{S}{d-1})$ is $(d-1)$-volume-rigid.
Union of volume-rigid complexes. Let K and L be $(d-1)$-volume-rigid complexes such that $|V(K) \cap V(L)| \ge d$. Then $K \cup L$ is $(d-1)$-volume-rigid.

Volume-rigid surfaces

Every triangulation of the 2-sphere, the torus, the projective plane or the Klein bottle is volume-rigid. In addition, every triangulation of the 2-sphere and the torus minus a single triangle is also volume-rigid. In particular, every simplicial disc with a 3-vertex boundary is minimally volume-rigid.
To show the claim for the torus, the projective plane and the Klein bottle we have verified by computer that the respective minimal triangulations are volume-rigid.

Volume-rigidity and sparsity

A $(d-1)$-complex is $(d-1, d^2-d-1)$-**sparse** (resp. **tight**) if every subset A of its vertices of cardinality at least d spans at most $(d-1)|A| - (d^2-d-1)$ simplices of dimensions $d-1$ (resp. and equality holds when A equals the entire vertex set).
Corollary. For every $d \ge 3$, there exists a $(d-1, d^2-d-1)$-tight $(d-1)$-complex that is not volume-rigid.

References

[1] Bulavka, D., Nevo, E., Peled, Y. *Volume rigidity and algebraic shifting*, ArXiv e-prints, 1810.11694 , 2018.
[2] Lubetzky, E. and Peled, Y. *The Threshold for Stacked Triangulations*, International Mathematics Research Notices, 2022. https://doi.org/10.1093/imrn/rnac276.

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