The distribution of descents on nonnesting permutations

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Background

Let $\pi = \pi_1\pi_2\dots\pi_m$ be a sequence of positive integers. We say that i is a

- **descent** if $\pi_i > \pi_{i+1}$ or $i = m$,
- **ascent** if $\pi_i < \pi_{i+1}$ or $i = 0$,
- **plateau** if $\pi_i = \pi_{i+1}$.

$\operatorname{des}(\pi)$ = number of descents of π
$\operatorname{asc}(\pi)$ = number of ascents of π
$\operatorname{plat}(\pi)$ = number of plateaus of π
$\operatorname{wdes}(\pi) = \operatorname{des}(\pi) + \operatorname{plat}(\pi)$ = number of weak descents of π

The **Eulerian polynomials**

$$A_n(t) = \sum_{\pi \in \mathcal{S}_n} t^{\operatorname{des}(\pi)}.$$

appear as numerators of the series

$$\sum_{m\geq 0} m^n t^m = \frac{A_n(t)}{(1-t)^{n+1}}.$$

Their exponential generating function is

$$A(t,z) = \sum_{n\geq 0} A_n(t)\frac{z^n}{n!} = \frac{1-t}{1-te^{(1-t)z}}.$$

Stirling permutations

Consider the multiset $[n] \sqcup [n] := \{1,1,2,2,\dots,n,n\}$.

Definition (Gessel–Stanley [5])
A **Stirling permutation** is a permutation $\pi_1\pi_2\dots\pi_{2n}$ of $[n]\sqcup[n]$ that avoids the pattern 212, i.e., there do not exist $i<j<k$ such that $\pi_i = \pi_k > \pi_j$.
$\mathcal{Q}_n$ = set of Stirling permutations of $[n]\sqcup[n]$.
Define the **Stirling polynomials**

$$Q_n(t) = \sum_{\pi\in\mathcal{Q}_n} t^{\operatorname{des}(\pi)}.$$

Example: $13324421 \in \mathcal{Q}_4$, but $312321 \notin \mathcal{Q}_3$.

Theorem (Gessel–Stanley [5])

$$\sum_{m\geq 0} S(m+n,m)\, t^m = \frac{Q_n(t)}{(1-t)^{2n+1}},$$

where $S(\ ,\ )$ denotes the Stirling numbers of the second kind.

Theorem (Bóna [2])

- On average, permutations in $\mathcal{Q}_n$ have $\frac{2n+1}{3}$ ascents, $\frac{2n+1}{3}$ descents and $\frac{2n+1}{3}$ plateaus.
- The distribution of the number of descents on $\mathcal{Q}_n$ is asymptotically normal.

Gessel and Stanley also defined ***k*-Stirling permutations** as permutations of the multiset $\{1^k, 2^k, \dots, n^k\}$ that avoid the pattern 212.

Noncrossing permutations

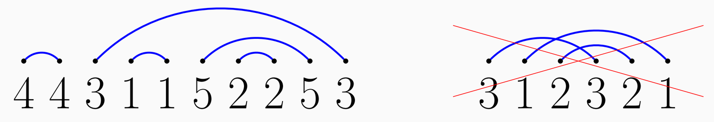
Definition (Archer et al. [1])
A **noncrossing** (or **quasi-Stirling**) **permutation** is a permutation $\pi_1\pi_2\dots\pi_{2n}$ of $[n]\sqcup[n]$ that avoids the patterns 1212 and 2121, i.e., there do not exist $i<j<k<\ell$ with $\pi_i=\pi_k$ and $\pi_j=\pi_\ell$.
$\overline{\mathcal{Q}}_n$ = set of noncrossing permutations of $[n]\sqcup[n]$.
Define the **quasi-Stirling polynomials**

$$\overline{Q}_n(t) = \sum_{\pi\in\overline{\mathcal{Q}}_n} t^{\operatorname{des}(\pi)}$$

and their exponential generating function

$$\overline{Q}(t,z) = \sum_{n\geq 0}\overline{Q}_n(t)\frac{z^n}{n!}.$$

$4431152253 \in \overline{\mathcal{Q}}_5$ $\qquad$ $312321 \notin \overline{\mathcal{Q}}_3$



Viewed as labeled noncrossing matchings,

$$|\overline{\mathcal{Q}}_n| = n!\operatorname{Cat}_n = \frac{(2n)!}{(n+1)!}.$$

Main theorem ([4])

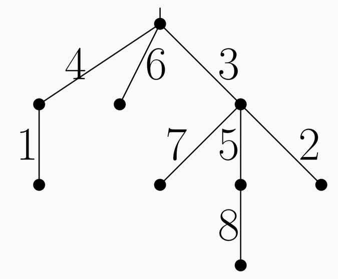
$\overline{Q}(t,z)$ satisfies the implicit equation

$$\overline{Q}(t,z) = A(t, z\overline{Q}(t,z)) = \frac{1-t}{1-te^{(1-t)z\overline{Q}(t,z)}}.$$

Its coefficients satisfy

$$\overline{Q}_n(t) = \frac{n!}{n+1}[z^n]A(t,z)^{n+1}.$$

The proof uses a bijection to edge-labeled plane rooted trees:



$\longleftrightarrow$ 4114663775885223

Corollary ([4], conjectured in [1])

The number of $\pi\in\overline{\mathcal{Q}}_n$ with $\operatorname{des}(\pi)=n$ is equal to $(n+1)^{n-1}$.

We also have a direct bijective proof.

Corollary ([4])

$$\sum_{m\geq 0}\frac{m^n}{n+1}\binom{m+n}{m}t^m = \frac{\overline{Q}_n(t)}{(1-t)^{2n+1}}$$

Theorem ([4])

- On average, permutations in $\overline{\mathcal{Q}}_n$ have $\frac{3n+1}{4}$ ascents, $\frac{3n+1}{4}$ descents and $\frac{n+1}{2}$ plateaus.
- The coefficients of $\overline{Q}_n(t)$ are unimodal and log-concave.
- The distribution of the number of descents on $\overline{\mathcal{Q}}_n$ is asymptotically normal.

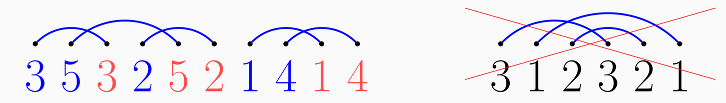
Most of these results extend to ***k*-quasi-Stirling permutations**, which are permutations of the multiset $\{1^k, 2^k, \dots, n^k\}$ that avoid the patterns 1212 and 2121.

Nonnesting permutations

Definition
A **nonnesting** (or **canon**) **permutation** is a permutation $\pi_1\pi_2\dots\pi_{2n}$ of $[n]\sqcup[n]$ that avoids the patterns 1221 and 2112, i.e., there do not exist $i<j<k<\ell$ with $\pi_i=\pi_\ell$ and $\pi_j=\pi_k$.
$\mathcal{C}_n$ = set of nonnesting permutations of $[n]\sqcup[n]$.
Consider the polynomials

$$C_n(t,u) = \sum_{\pi\in\mathcal{C}_n} t^{\operatorname{des}(\pi)}u^{\operatorname{plat}(\pi)}.$$

$3532521414 \in \mathcal{C}_5$ $\qquad$ $312321 \notin \mathcal{C}_3$

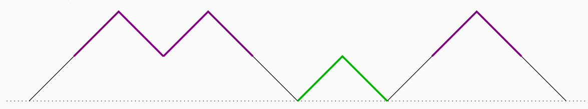


Viewed as labeled nonnseting matchings,

$$|\mathcal{C}_n| = n!\operatorname{Cat}_n = \frac{(2n)!}{(n+1)!}.$$

π is nonnesting $\iff$ subsequence of first copies = subsequence of second copies

A peak in a Dyck path $D\in\mathcal{D}_n$ is low if it touches the x-axis, and high otherwise.

Example:  has 1 low peak and 3 high peaks.

The **Narayana polynomials**

$$N_n(t,u) = \sum_{D\in\mathcal{D}_n} t^{\#\text{high peaks of } D}u^{\#\text{low peaks of } D}$$

have a well-known generating function

$$\sum_{n\geq 0} N_n(t,u)z^n = \frac{2}{1+(1+t-2u)z+\sqrt{1-2(1+t)z+(1-t)^2z^2}}.$$

Main theorem ([3])

$$C_n(t,u) = A_n(t)\,N_n(t,u).$$

As a consequence, since the polynomials $A_n(t)$, $N_n(t,1)$ and $N_n(t,t)$ are palindromic, so are $C_n(t,1) = A_n(t)N_n(t,1)$ and $C_n(t,t) = A_n(t)N_n(t,t)$.

Corollary ([3])

The distributions of descents and weak descents on $\mathcal{C}_n$ are symmetric: for all r,

$$|\{\pi\in\mathcal{C}_n : \operatorname{des}(\pi)=r\}| = |\{\pi\in\mathcal{C}_n : \operatorname{des}(\pi)=2n-r\}|,$$
$$|\{\pi\in\mathcal{C}_n : \operatorname{wdes}(\pi)=r\}| = |\{\pi\in\mathcal{C}_n : \operatorname{wdes}(\pi)=2n+2-r\}|.$$

We have bijective proofs of these symmetries but they are surprisingly complicated!

References

[1] Kassie Archer, Adam Gregory, Bryan Pennington, and Stephanie Slayden. "Pattern restricted quasi-Stirling permutations". In: *Australas. J. Combin.* 74 (2019), pp. 389–407.
[2] Miklós Bóna. "Real zeros and normal distribution for statistics on Stirling permutations defined by Gessel and Stanley". In: *SIAM J. Discrete Math.* 23.1 (2008), pp. 401–406.
[3] Sergi Elizalde. "Descents on nonnesting multipermutations". preprint, arXiv:2204.00165.
[4] Sergi Elizalde. "Descents on quasi-Stirling permutations". In: *J. Combin. Theory Ser. A* 180 (2021), Paper No. 105429, 35.
[5] Ira M. Gessel and Richard P. Stanley. "Stirling polynomials". In: *J. Combin. Theory Ser. A* 24.1 (1978), pp. 24–33.

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