A combinatorial interpretation of the NSym inverse Kostka matrix

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The classical inverse Kostka matrix

- $h_k = \sum_{1 \le i_1 \le \cdots \le i_k} x_{i_1} \cdots x_{i_k}, \qquad h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_\ell}$
- **Jacobi-Trudi formula:** If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, then
$$s_\lambda = \det(h_{\lambda_i + j - i})_{i,j}.$$
- The *inverse Kostka matrix* K^{-1} is the transition matrix from $\{s_\lambda\}_{\lambda \vdash n}$ to $\{h_\lambda\}_{\lambda \vdash n}$ in *Sym*.
- **Eğecioğlu and Remmel [4]**: Fill partition diagrams with "special rim hooks" and assign signed weights to these fillings to obtain the inverse Kostka matrix entries.

Immaculate functions $\mathfrak{S}_\mu$ [2]

- $\{H_1, H_2, \ldots\}$ = algebraically independent functions that don't commute; for any composition $\alpha = (\alpha_1, \ldots, \alpha_\ell)$,
$$H_\alpha = H_{\alpha_1} H_{\alpha_2} \cdots H_{\alpha_\ell}.$$
- *NSym* is generated by $H_1, H_2, \ldots$ (with no relations).
- Let M_μ be the matrix $(M_\mu)_{i,j} = H_{\mu_i + j - i}$. Then
$$\mathfrak{S}_\mu = \mathfrak{det}(M_\mu),$$
where $\mathfrak{det}$ is the *NSym* determinant.

Main goal and results:

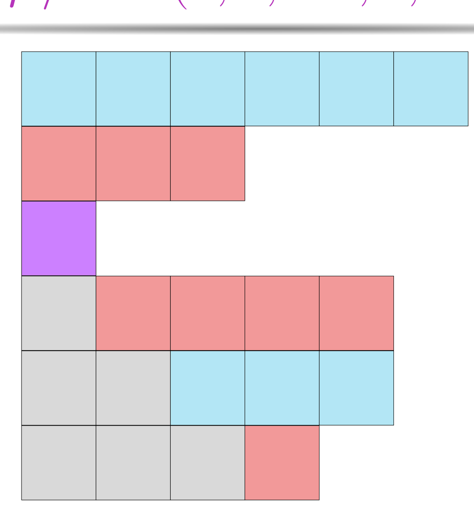
The main goal of this project is to generalize the Eğecioğlu-Remmel construction to the *NSym* setting. We provide:

- a diagram filling construction to compute the H-basis decomposition of the immaculate functions,
- an extension to skew immaculates, and
- new results on the ribbon decomposition of immaculate functions.

GBPR Diagram (sequence μ, partition λ)

1. Place λ_i grey cells in row i (bottom to top)
2. For each $1 \le i \le k$:
   1. If $\mu_i > 0$ and $\lambda_i \le \mu_i$, append $\mu_i - \lambda_i$ blue cells to row i.
   2. If $\mu_i > 0$ and $\mu_i < \lambda_i$, append $\lambda_i - \mu_i$ red cells to row i.
   3. If $\mu_i \le 0$, append $|\mu_i| + \lambda_i$ red cells to row i.
3. All other cells are purple.

Example: $\mu/\lambda = (2, 5, -3, 0, -3, 6)/(3, 2, 1)$



If μ is a partition, the GBPR diagram is an ordinary skew partition diagram. Furthermore, the forgetful map (which "forgets" that the H functions don't commute) implies that the constructions in this project are valid in both *Sym* and *NSym*.

Theorem [A-M, 2022]

For $\mu \in \mathbb{Z}^k$,
$$\mathfrak{S}_\mu = \sum_{\gamma \in \mathrm{THC}_\mu} \prod_{r=1}^{k} \epsilon(\mathfrak{h}(r, \tau_r))\, H_{\Delta(\mathfrak{h}(r,\tau_r))}, \quad \text{where}$$

- THC_μ = the set of tunnel hook coverings of a GBPR diagram of shape μ,
- $\mathfrak{h}(r, \tau_r)$ is a tunnel hook starting in row r of a THC,
- $\epsilon(\mathfrak{h}(r, \tau_r))$ is the sign of $\mathfrak{h}(r, \tau_r)$, and
- $\Delta(\mathfrak{h}(r, \tau_r))$ is a value assigned to tunnel hook $\mathfrak{h}(r, \tau_r)$.

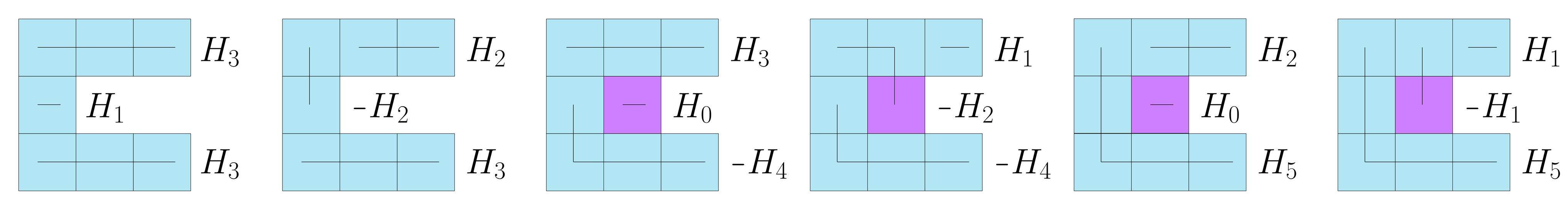
Tunnel hook coverings and weights

A *tunnel hook* $\mathfrak{h}(r, \tau_r)$ of height p includes all cells in row r as well as some collection of boundary cells (adjacent to grey cells on left) in higher rows, spanning a total of p rows.

- $\Delta(\mathfrak{h}(r, \tau_r))$ = (blue cells - red cells) in row r + taxicab distance to end of tunnel hook
- $\epsilon(\mathfrak{h}(r, \tau_r)) = (-1)^{p+1}$
- A *tunnel hook covering* (THC) in THC_μ is a covering of the GBPR diagram of μ with non-overlapping tunnel hooks, one starting in each row.

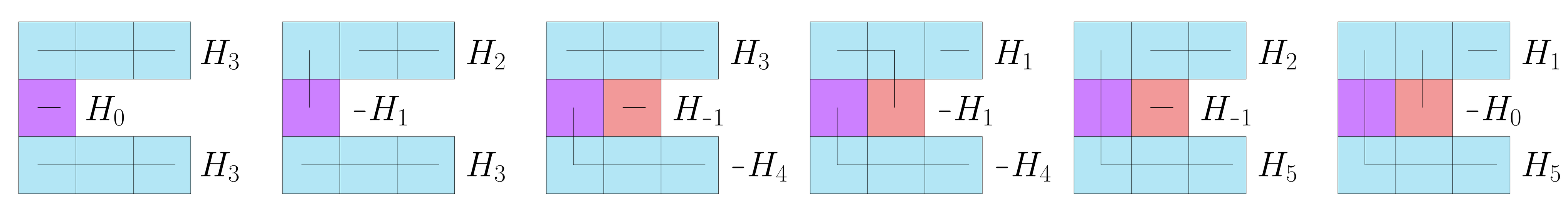
A trio of examples

$\mu = (3, 1, 3)$



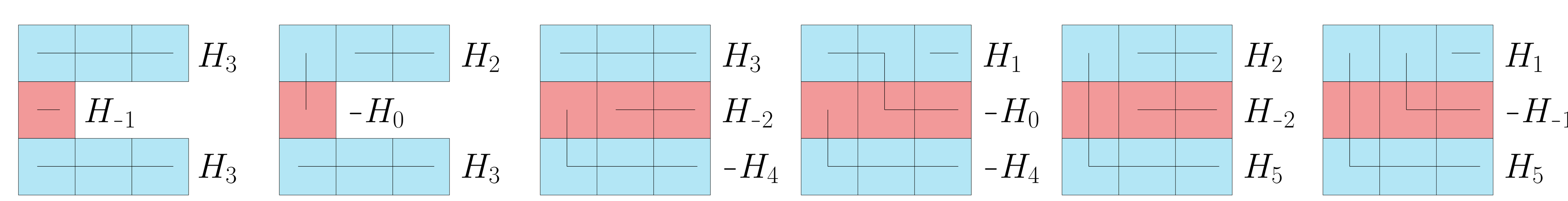
$$\mathfrak{S}_{(3,1,3)} = H_{(3,1,3)} - H_{(3,2,2)} - H_{(4,3)} + H_{(4,2,1)} + H_{(5,2)} - H_{(5,1,1)}$$

$\mu = (3, 0, 3)$



$$\mathfrak{S}_{(3,0,3)} = H_{(3,3)} - H_{(3,1,2)} + H_{(4,1,1)} - H_{(5,1)}$$

$\mu = (3, -1, 3)$

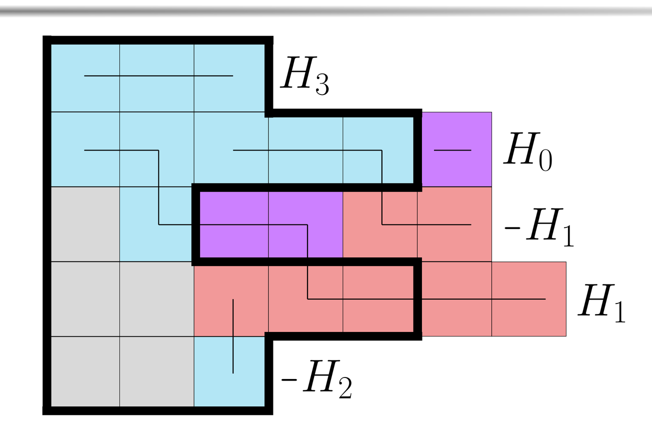


$$\mathfrak{S}_{(3,-1,3)} = -H_{(3,2)} + H_{(4,1)}$$

Skew immaculate functions $\mathfrak{S}_{\mu/\nu}$

Let μ and ν be arbitrary sequences of integers and $M_{\mu/\nu}$ be the matrix $(M_{\mu/\nu})_{i,j} = H_{\mu_i - i - (\nu_j - j)}$. Then
$$\mathfrak{S}_{\mu/\nu} = \mathfrak{det}(M_{\mu/\nu}),$$
where $\mathfrak{det}$ is the *NSym* determinant.

- This definition generalizes the Jacobi-Trudi formula for skew Schur functions to *NSym*.
- Our tunnel hook construction extends to skewing by a partition.



$$\begin{bmatrix} H_1 & \boxed{H_2} & H_4 & H_6 & H_7 \\ H_{-4} & H_{-3} & H_{-1} & \boxed{H_1} & H_2 \\ H_{-2} & H_{-1} & \boxed{H_1} & H_3 & H_4 \\ \boxed{H_0} & H_1 & H_3 & H_5 & H_6 \\ H_{-3} & H_{-2} & H_0 & H_2 & \boxed{H_3} \end{bmatrix}$$

Example: $\mu = (3, -1, 2, 5, 3), \lambda = (2, 2, 1)$

$$\mathfrak{S}_{\mu/\lambda} = \mathfrak{det} \begin{bmatrix} H_1 & H_2 & H_4 & H_6 & H_7 \\ H_{-4} & H_{-3} & H_{-1} & H_1 & H_2 \\ H_{-2} & H_{-1} & H_1 & H_3 & H_4 \\ H_0 & H_1 & H_3 & H_5 & H_6 \\ H_{-3} & H_{-2} & H_0 & H_2 & H_3 \end{bmatrix}$$
$$= -H_{(1,1,1,1,3)} + H_{(1,1,4,1,0)} + H_{(1,2,1,1,2)} - H_{(1,2,3,1,0)} + H_{(2,1,1,0,3)} - H_{(2,1,4,0,0)} - H_{(2,2,1,0,2)} + H_{(2,2,3,0,0)}$$

Theorem [A-M, 2022]

For $\mu \in \mathbb{Z}^k$ and partition λ,
$$\mathfrak{S}_\mu = \sum_{\gamma \in \mathrm{THC}_{\mu/\lambda}} \prod_{r=1}^{k} \epsilon(\mathfrak{h}(r, \tau_r))\, H_{\Delta(\mathfrak{h}(r,\tau_r))}, \quad \text{where}$$

- $\mathrm{THC}_{\mu/\lambda}$ = the set of tunnel hook coverings of a GBPR diagram of shape μ,
- $\mathfrak{h}(r, \tau_r)$ is a tunnel hook starting in row r of a THC,
- $\epsilon(\mathfrak{h}(r, \tau_r))$ is the sign of $\mathfrak{h}(r, \tau_r)$, and
- $\Delta(\mathfrak{h}(r, \tau_r))$ is a value assigned to tunnel hook $\mathfrak{h}(r, \tau_r)$.

Decomposition into ribbon basis

The *ribbon basis* for *NSym* is dual to the fundamental basis for quasisymmetric functions (*QSym*).
$$R_\alpha = \sum_{\beta \succeq \alpha} (-1)^{\ell(\beta) - \ell(\alpha)} H_\beta$$

- **Open Problem:** Find a formula for the decomposition of the immaculate functions into the ribbon basis.
- **Rectangles**: If $\alpha = (m^k)$, then the indices appearing in the ribbon expansion equal the length k indices appearing in the H-basis expansion [3].
- **Example:** $\alpha = (2, 2, 2)$
$\mathfrak{S}_{(2,2,2)} = H_{(2,2,2)} - H_{(2,3,1)} - H_{(3,1,2)} + H_{(4,1,1)} + H_{(3,3)} - H_{(4,2)}$
$\mathfrak{S}_{(2,2,2)} = R_{(2,2,2)} - R_{(2,3,1)} - R_{(3,1,2)} + R_{(4,1,1)}$
- We use our tunnel hook construction to extend this result to a larger (but not complete) class of compositions.

Theorem [A-M, 2022]

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ be a composition such that for some $1 \le j \le \ell$,

1. $\alpha_i \ge i$ for $1 \le i \le j$, and
2. $\alpha_{j+1} = j$, and
3. $\alpha_{j+1} = \alpha_{j+2} = \ldots = \alpha_\ell$.

Then
$$\mathfrak{S}_\alpha = \sum_{\sigma \in S_k} \epsilon(\sigma) R_{(\alpha_1 - 1 + \sigma_1, \alpha_2 - 2 + \sigma_2, \ldots, \alpha_k - k + \sigma_k)},$$
with the convention that R_α vanishes if α contains any nonpositive parts.

Further directions

- Understand the relationship between skew immaculate functions and products of dual immaculate functions.
- Use tunnel hook coverings to determine the ribbon expansion of an arbitrary immaculate function.
- Extend this expansion to other bases for *NSym* and *QSym* such as the Young quasisymmetric Schur functions.

For Further Information

1. Allen, E. and Mason, S. A combinatorial interpretation of the noncommutative inverse Kostka matrix. *arXiv:2207.05903* (2022).
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3. Campbell, J. The expansion of immaculate functions in the ribbon basis. *Discrete Math*, 340 (2017) 7:1716-1726.
4. Eğecioğlu, Ö. and Remmel, J. A combinatorial interpretation of the inverse Kostka matrix. *Linear and Multilinear Algebra*, 26 (1990) 1-2:59-84.
5. Gelfand, I., Krob, D., Lascoux, A., Leclerc, B., Retakh, V., and Thibon, J.Y. Noncommutative symmetric functions. *Adv. Math.*, 112 (1995) 2:218-348.
6. Loehr, N. and Niese, E. Combinatorics of the immaculate inverse Kostka matrix. *Algebraic Combinatorics*, 4 (2021) 6: 1119-1142.