Generic Curves and non-coprime Catalans

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Outline

We compute the Poincaré polynomials of the compactified Jacobians for plane curve singularities

$$x = t^{nd}, y = t^{md} + \lambda t^{md+1} + \ldots,\ \lambda \neq 0$$

and relate them to the q, t-Catalan combinatorics in the non-coprime case.

Compactified Jacobians and Semigroups

Let C be a plane curve singularity parametrized by $(x(t), y(t))$, and let $\mathcal{O}_C = \mathbb{C}[[x(t), y(t)]] \subset \mathbb{C}[[t]]$ be the local ring of functions on C. For $f(t) \in \mathcal{O}_C$, write

$$f(t) = \alpha_k t^k + \alpha_{k+1} t^{k+1} + \ldots,\ \alpha_k \neq 0$$

and set $\mathrm{Ord} f(t) := k$. The compactified Jacobian $\overline{JC}$ is defined as the moduli space of $\mathcal{O}_C$-submodules $M \subset \mathbb{C}[[t]]$. Note that M is an $\mathcal{O}_C$–submodule if and only if $x(t)M \subset M,\ y(t)M \subset M$. Given such a subspace M, we define

$$\Delta_M = \{\mathrm{Ord}\ f(t)\ :\ f(t) \in M\} \subset \mathbb{Z}_{\geq 0}.$$

In particular, for $M = \mathcal{O}_C$ we obtain the **semigroup** of C:

$$\Gamma = \Delta_{\mathcal{O}_C} = \{\mathrm{Ord}\ f(t)\ :\ f(t) \in \mathcal{O}_C\}.$$

If M is an $\mathcal{O}_C$-submodule then Δ_M is a Γ-module: $\Delta_M + \Gamma \subset \Delta_M$.

Definition

Given a subset $\Delta \subset \mathbb{Z}_{\geq 0}$, we define the stratum in the compactified Jacobian:

$$J_\Delta := \{M \subset \mathbb{C}[[t]] : \mathcal{O}_C M \subset M,\ \Delta_M = \Delta\}.$$

Clearly, the strata J_Δ give a subdivision of $\overline{JC}$ and J_Δ is empty if Δ is not Γ-invariant. However, J_Δ could be empty even if Δ is Γ-invariant.

Example

Consider the curve $(x(t), y(t)) = (t^4, t^6 + t^7)$. It is easy to see that $y^2(t) - x^3(t) = 2t^{13} + t^{14}$ has order 13, and one can check that the semigroup Γ in this case is generated by 4, 6 and 13. The subset

$$\Delta = \{0, 2, 4, 6, 8, 10, 12, 13, 14, \ldots\}$$

is clearly Γ-invariant. Suppose that $M \subset J_\Delta$, and $f_0 = 1 + at + \ldots$ and $f_2 = t^2 + bt^3 + \ldots$ are two elements of M of orders 0 and 2 respectively. Then

$$y(t)f_0 - x(t)f_2 = (t^6 + t^7)(1 + at + \ldots) - t^4(t^2 + bt^3 + \ldots) = (a + 1 - b)t^7 + \ldots,$$

$$x^2(t)f_0 - y(t)f_2 = t^8(1 + at + \ldots) - (t^6 + t^7)(t^2 + bt^3 + \ldots) = (a - b - 1)t^9 + \ldots$$

Observe that $a + 1 - b$ and $a - b - 1$ cannot vanish at the same time, so either M contains an element of order 7 or an element of order 9, which is a contradiction, since $7 \notin \Delta$ and $9 \notin \Delta$. Therefore the corresponding stratum J_Δ is empty.

Any unibranched planar curve C can be parametrized using Puiseaux expansion:

$$x = t^{nd},\ y = t^{md} + \lambda t^{md+1} + \ldots,\ \mathrm{GCD}(m, n) = 1.$$

If $d = 1$ then the semigroup of C is generated by m and n and the link of C is the (n, m) torus knot.
Assume that $d > 1$. Then a curve C is called a ***generic curve*** if $\lambda \neq 0$. It is easy to see that the definition of a generic curve is symmetric in n and m. The semigroup Γ of a generic curve is generated by dn, dm and $dnm + 1$. The corresponding knot as a $(d, mnd + 1)$-cable of the (n, m) torus knot.

Invariant Subsets and the Cell Decomposition

We call a subset $\Delta \subset \mathbb{Z}_{\geq 0}$ an *(nd)-invariant subset* if $nd + \Delta \subset \Delta$. A number a is called an *(nd)-generator* of Δ if $a \in \Delta$ but $a - nd \notin \Delta$. Let A denote the set of all (nd)-generators of Δ.
For any element $b \in \Delta$ let us use the notation

$$\mathrm{Gaps}(b) := \{c \in \mathbb{Z} | c > b,\ c \notin \Delta\}.$$

Furthermore, set

$$G(\Delta) := \sum_{a \in A} |\mathrm{Gaps}(a)| \quad \text{and} \quad E(\Delta) := \sum_{a \in A} |\mathrm{Gaps}(a + dm)|.$$

We show that J_Δ is an affine subvariety inside $\mathbb{C}^{G(\Delta)}$ defined by $E(\Delta)$ equations. However, as we saw in the Example above, sometimes the equations have no solutions.
It is convenient to split Δ into d pieces according to the remainders modulo d. We write

$$\Delta_j := \Delta \cap (d\mathbb{Z} + j), \quad \Delta = \bigcup_{j=0,\ldots,d-1} \Delta_j.$$

Definition

We call Δ admissible, if for every j one has $\Delta_{j+1} \not\subset \Delta_j + md + nd + 1$.

We prove that for an inadmissible subset Δ the set of equations defined above is inconsistent, while for an admissible subset Δ the equations are essentially independent and can be used to eliminate parameters:

Theorem

$J_\Delta = \emptyset$ unless Δ is admissible, in which case

$$J_\Delta = \mathbb{C}^{\dim(\Delta)}, \quad \dim(\Delta) = G(\Delta) - E(\Delta).$$

Continuing the Example above, one gets:

Example

We have $n = 2$, $m = 3$, and $d = 2$. For the subset

$$\Delta = \{0, 2, 4, 6, 8, 10, 12, 13, 14, \ldots\}$$

one gets

$$\Delta_0 = \{0, 2, 4, 8, \ldots\} = 2\mathbb{Z}_{\geq 0} \quad \text{and} \quad \Delta_1 = \{13, 15, \ldots\} = 2\mathbb{Z}_{\geq 0} + 13.$$

One gets $nd + md + 1 = 4 + 6 + 1 = 11$ and

$$\Delta_1 \subset \Delta_0 + nd + md + 1.$$

Hence Δ is inadmissible.

Note that even though the semigroup Γ of a generic curve is generated by dn, dm, and $dnm + 1$, one doesn't have to require Δ to be $(dnm + 1)$-invariant. This is due to the following elementary observation:

Lemma

Any admissible $(nd, md)-$invariant subset is also $(dnm + 1)-$invariant.

Combinatorics

Let $\mathrm{Inv}(dm, dn)$ be the set of all cofinite 0-normalized (dm, dn)-invariant subsets:

$$\mathrm{Inv}(dm, dn) := \{\Delta \in \mathbb{Z}_{\geq 0} | 0 \in \Delta,\ \Delta + dn \subset \Delta,\ \Delta + dm \subset \Delta,\ |\mathbb{Z}_{\geq 0} \setminus \Delta| < \infty\}$$

Let also $\mathrm{Dyck}(dm, dn)$ be the set of all (dn, dm) Dyck paths, i.e. south-west lattice paths connecting $(0, dn)$ to $(dm, 0)$ and staying (weakly) under the diagonal.
In [2] the first two authors and Monica Vazirani defined an equivalence relation $\sim$ on $\mathrm{Inv}(dm, dn)$ and a bijection

$$\mathcal{D} : \mathrm{Inv}(dm, dn)/\sim \to \mathrm{Dyck}(dm, dn),$$

such that $\dim \Delta = \delta - \mathtt{dinv}(\mathcal{D}(\Delta))$ for all $\Delta \in \mathrm{Inv}(nd, md)$.
We prove the following result:

Theorem

Every equivalence class contains a unique admissible representative.

In particular, one gets the following simple formula for the Poincaré polynomial of the Compactified Jacobian:

Corollary

The Poincaré polynomial of $\overline{JC}$ is given by

$$P_{\overline{JC}}(t) = \sum_{D \in \mathrm{Dyck}(nd, md)} t^{2(\delta - \mathtt{dinv}(D))}$$

The rectangular q, t-Catalan is defined by

$$C_{nd,md}(q, t) = \sum_{D \in \mathrm{Dyck}(dn, dm)} q^{\mathtt{area}(D)} t^{\mathtt{dinv}(D)}.$$

The Compositional Rational Shuffle Theorem, conjectured in [1] and proved in [4], implies that $C_{nd,md}(q, t)$ is symmetric in q and t. In particular, one gets an even simplier formula for the Poincaré polynomial of $\overline{JC}$:

Corollary

The Poincaré polynomial of $\overline{JC}$ is given by

$$P_{\overline{JC}}(t) = \sum_{D \in \mathrm{Dyck}(nd, md)} t^{2(\delta - \mathtt{area}(D))}$$

Bibliography

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