A cocharge formula for the Δ-Springer modules

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Main Theorem (G., G., 2023)

Let $R_{n,\lambda,s}$ be a Δ-*Springer module*. We have the Schur expansion

$$\widetilde{H}_{n,\lambda,s}(x;q) := \mathrm{grFrob}(R_{n,\lambda,s}) = \frac{1}{q^{\binom{s-1}{2}(n-k)}} \sum_{T \in \mathcal{T}^+(n,\lambda,s)} q^{\mathrm{cc}(T)} s_{\mathrm{sh}^+(T)}(x)$$

where $\mathcal{T}^+$ is a set of *battery-powered tableaux* and cc is the *cocharge* statistic.

Background: What is a Δ-Springer module?

Ingredients to make a Δ-Springer module $R_{n,\lambda,s}$

1. A positive integer n
2. A partition λ of size $k := |\lambda| < n$
3. A number $s > \ell(\lambda)$

Specializations:

Delta: $R_{n,(1^k),k} = R_{n,k}$

Springer: $R_{n,\mu,\ell(\mu)} = R_\mu$ for $\mu \vdash n$

Recipe: Define $I_{n,\lambda,s} = (x_1^s, \ldots, x_n^s, e_r(S))$ for certain partial elementary symmetric functions $e_r(S)$; invariant under S_n action on variables.

Output: The graded S_n-module

$$R_{n,\lambda,s} := \mathbb{Q}[x_1, \ldots, x_n]/I_{n,\lambda,s}.$$

Background: Charge and cocharge on words

Charge of a standard word: Label the 1 with a *charge subscript* 0, then label $i = 2, 3, 4, \ldots, n$ where subscript is incremented if i is right of $i-1$:

$$6257134 \to 6_2 2_0 5_2 7_3 1_0 3_1 4_2 \qquad \mathrm{ch}(6257134) = 2+0+2+3+0+1+2 = 10$$

Cocharge of a standard word: Increment subscripts if i is *left* of $i-1$:

$$6257134 \to 6_3 2_1 5_2 7_3 1_0 3_1 4_1 \qquad \mathrm{cc}(6257134) = 3+1+2+3+0+1+1 = 11$$

Subwords: If w is a general word with partition content, to form its *first charge subword* $w^{(1)}$, search from the right to find a $1, 2, 3, \ldots$, wrapping around the end cyclically if need be:

$$w = 213413122 \qquad w^{(1)} = 2\_\_4\_31\_\_$$

Remove $w^{(1)}$ and repeat to find the second cocharge subword $w^{(2)}$, etc.

Charge/cocharge of a word w with partition content:

$$\mathrm{cc}(w) = \sum \mathrm{cc}(w^{(i)}) \qquad \mathrm{ch}(w) = \sum \mathrm{ch}(w^{(i)})$$

Background: Graded Frobenius series

Recall $\mathrm{Frob}(V_\lambda) = s_\lambda$ where s_λ is a Schur function and V_λ is the irreducible S_n representation corresponding to λ. Also have

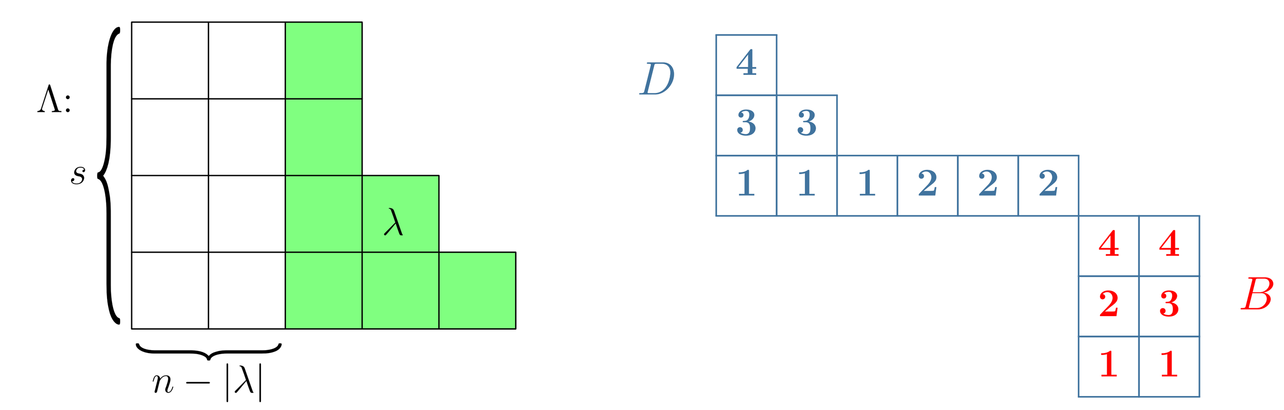
$$\mathrm{Frob}(V \oplus W) = \mathrm{Frob}(V) + \mathrm{Frob}(W).$$

Graded Frobenius: If $R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$ is a graded ring,

$$\mathrm{grFrob}_q(R) = \sum_d q^d \mathrm{Frob}(R_d)$$

New: Battery-powered tableaux!

Say $n = 9$, $\lambda = (3,2,1,1)$, $s = 4$. Define $\Lambda = (n-k)^s + \lambda$ as shown.



Device: D, semistandard, $|D| = n$

Battery: B, semistandard, $(s-1) \times (n-k)$ rectangle

Total content of (D,B) is Λ: The entry i appears Λ_i times

Battery-powered tableau of parameters n, λ, s - a pair $T = (D,B)$ as above. Write $\mathcal{T}^+(n,\lambda,s)$ for the set of battery-powered tableaux.

Cocharge: $\mathrm{cc}(T) = \mathrm{cc}(w)$ where w is formed by reading the rows of D and then of B from top to bottom. *Can you compute the above tableau's cocharge?*

Shape: $\mathrm{sh}^+(T) = \mathrm{sh}(D)$. Above, shape is $(6,2,1)$.

Charge version of Main Theorem

Define rev_q of a polynomial by setting $q \mapsto q^{-1}$ and multiplying through by the highest power of q. Then:

$$\mathrm{rev}_q\left(\widetilde{H}_{n,\lambda,s}\right) = \mathrm{rev}_q\left(\mathrm{grFrob}(R_{n,\lambda,s})\right) = \sum_{T \in \mathcal{T}^+(n,\lambda,s)} q^{\mathrm{ch}(T)} s_{\mathrm{sh}^+(T)}(x).$$

Specialization to "Delta" case

If $\lambda = 1^k$ and $s = k$, we have $R_{n,\lambda,s} = R_{n,k}$, the *Haglund-Rhoades-Shimozono modules*. The *Delta Conjecture* gives combinatorial expansions in two parameters q, t for $\Delta'_{e_{k-1}} e_n$ where $\Delta'_{e_{k-1}}$ is a certain Macdonald eigenoperator, and it is known that $\mathrm{grFrob}(R_{n,k}) = \omega \circ \mathrm{rev}_q\left(\Delta'_{e_{k-1}} e_n|_{t=0}\right)$.

Corollary. We have a new Schur expansion and skewing formula at $t = 0$:

$$\Delta'_{e_{k-1}} e_n|_{t=0} = \sum_{T \in \mathcal{T}^+(n,(1^k),k)} q^{\mathrm{ch}(T)} s_{\mathrm{sh}^+(T)^*}(x) = \omega \cdot s^\perp_{(n-k)^{k-1}} H_\Lambda(x;q).$$

Specialization to "Springer" case

If $k = n$, that is, $\lambda \vdash n$, then $R_{n,\lambda,s} = R_\lambda$, a *Garsia-Procesi module* whose Frobenius series is a Hall-Littlewood polynomial:

$$\mathrm{grFrob}(R_\lambda) = \widetilde{H}_\lambda(x;q) = \sum_{T \text{ content } \lambda} q^{\mathrm{cc}(T)} s_{\mathrm{sh}(T)}$$

Here $R_\lambda = H^*(\mathcal{B}_\lambda)$ where $\mathcal{B}_\lambda$ is a *Springer fiber*.

Background: Borho–Macpherson partial resolutions

Partial flag varieties: $G = \mathrm{GL}_K(\mathbb{C})$, P parabolic subgroup (block upper triangular), B borel (upper trianglar). Partial flag variety is G/P, complete is G/B.

Partial resolutions: $\mathcal{N}$ is cone of nilpotent matrices,

$$\widetilde{\mathcal{N}}^P := \{(n, F_\bullet) \mid F_\bullet \in G/P,\ nF_i \subseteq F_i\ \forall i,\ n \in \mathcal{N}\}$$

is rationally smooth, $\widetilde{\mathcal{N}} := \widetilde{\mathcal{N}}^B$ is smooth. Springer resolution $\widetilde{\mathcal{N}} \to \mathcal{N}$ factors:

$$\widetilde{\mathcal{N}} \xrightarrow{\eta} \widetilde{\mathcal{N}}^P \xrightarrow{\rho} \mathcal{N}.$$

Orbit closures: Let $y \in \mathcal{N}^P$ map to $t + u \in \mathcal{N}$ where t is a block diagonal nilpotent with blocks given by P, u block strictly upper triangular. Orbit $\mathcal{O}_y$ defined using adjoint action by a Levi subgroup, $\overline{\mathcal{O}_y}$ its closure.

Borho–Macpherson fibers: Let $x \in \mathcal{N}$, define $\mathcal{P}_x^y = \rho^{-1}(x) \cap \overline{\mathcal{O}_y}$. Concretely:

$$\mathcal{P}_x^y \cong \{F_\bullet \in G/P \mid xF_i \subseteq F_i \text{ and } \mathrm{JT}(x|_{F_i/F_{i-1}}) \preceq \mathrm{JT}(t_i) \text{ for all } i\}.$$

Proof part 1: Δ-Springer varieties as fibers

G. (second author), Levinson, Woo: Constructed a Δ-*Springer variety* $Y_{n,\lambda,s}$ such that $H^*(Y_{n,\lambda,s}) = R_{n,\lambda,s}$.

Proposition (G., G.): Let P such that flags in G/P has parts in dimensions $1, 2, \ldots, n, K = |\Lambda|$. Then $Y_{n,\lambda,s} = \mathcal{P}_x^y$ where x has Jordan type Λ and t has block sizes $1, 1, \ldots, 1, K - n$ with the last block having Jordan type $(n-k)^{s-1}$.

Theorem (G., G.): $\overline{\mathcal{O}_y}$ is rationally smooth at all points of $\mathcal{P}_x^y$ in this case.

Idea of proof: Combinatorics of q-Kostka polynomials give the intersection cohomology. This shows the rectangular battery is geometrically special.

Proof part 2: Skewing formula

Using the above connections and a theorem of Borho–Macpherson in the case where $\overline{\mathcal{O}}_y$ is rationally smooth at all points of the fiber, we find

$$q^{\binom{s-1}{2}(n-k)} \widetilde{H}_{n,\lambda,s}(x;q) = s^\perp_{((n-k)^{s-1})} \widetilde{H}_\Lambda(x;q)$$

where $s_\nu^\perp$ is the adjoint operation to multiplication by s_ν with respect to the Hall inner product, and where $\widetilde{H}_\Lambda(x;q)$ is a Hall-Littlewood polynomial.

Manipulating the above formula using symmetric function theory identities and combinatorics then proves the main theorem.

Towards a combinatorial proof

We have a direct combinatorial proof of the main theorem for:

- $s = 2$ and any n, λ,
- The coefficient of $s_{(n)}$ in the $t = 0$ Delta conjecture case
- The coefficient of $s_{(n)}$ when λ is 'wide'

A full combinatorial proof would be of interest!

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