Combinatorial interpretations of the Macdonald identities for affine root systems

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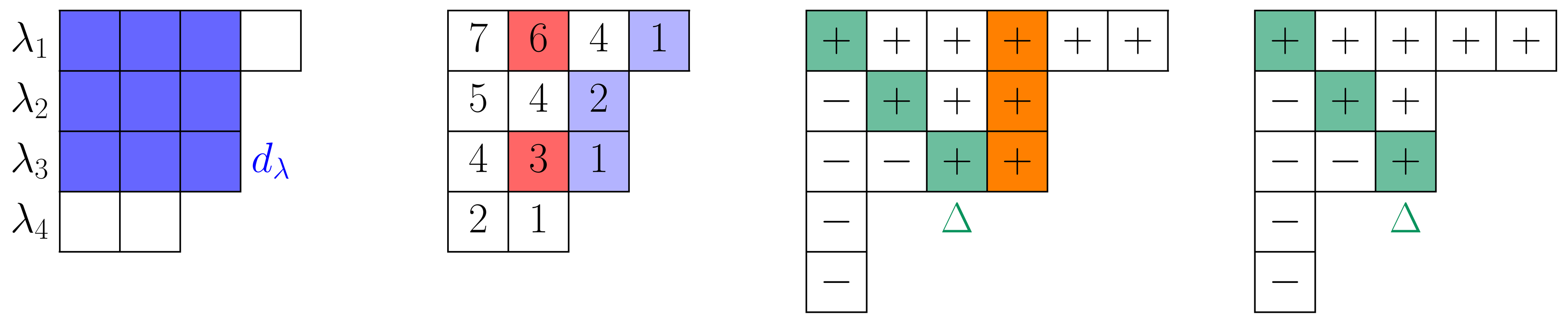
Introduction

- $\mathcal{P} := \{\text{integer partitions}\}$.
- $|\lambda| := \sum_{i=1}^{\ell}\lambda_i$.
- Conjugate λ' of λ : partition whose Ferrers diagram is obtained by the reflection of the Ferrers diagram of λ along the main diagonal.
- Self-conjugate partition $\lambda \in \mathcal{SC}$: $\lambda = \lambda'$.
- Doubled-distinct partition $\lambda \in \mathcal{DD}$: partition such that $\lambda = \lambda' + 1^{d_\lambda}$.
- t-core partition: no hook length divisible by $t \in \mathbb{N}^*$, $h(i,j) := \lambda_i + \lambda'_j - i - j + 1$ for $(i,j) \in \lambda$.
  $\mathcal{H}(\lambda) := \{\text{hook lengths of}\lambda\}$. $\mathcal{H}_t(\lambda) := \{h \in \mathcal{H}(\lambda) \mid h \equiv 0 \pmod t\}$ and $H_{+,t}(\lambda) := \{h \in \mathcal{H}(\lambda) \mid h < t, \varepsilon_h = 1\}$



(a) Durfee square (b) Hook lengths (c) $\lambda = (6,4,4,1,1) \in \mathcal{DD}$ (d) $\lambda = (5,3,3,1,1) \in \mathcal{SC}$

Goals

- Connection between Macdonald identities for the 7 infinite affine root systems and subsets of $\mathcal{P}$
- Enumerating hook lengths products of the previous subsets of $\mathcal{P}$
- Derive q-Nekrasov–Okounkov formulas for each of the 7 infinite types of affine root systems

Nekrasov–Okounkov formula

Nekrasov–Okounkov (2006), Westbury (2006), Han (2008)

$$\sum_{\lambda\in\mathcal{P}} T^{|\lambda|}\prod_{h\in\mathcal{H}(\lambda)}\left(1-\frac{z}{h^2}\right) = (T;T)_\infty^{z-1}$$

where $(a;T)_\infty := (1-a)(1-aT)(1-aT^2)\cdots$

A q-Nekrasov–Okounkov formula

Dehaye–Han (2011), Iqbal *et al.* (2012)

$$\sum_{\lambda\in\mathcal{P}} T^{|\lambda|}\prod_{h\in\mathcal{H}(\lambda)}\frac{(1-uq^h)(1-u^{-1}q^h)}{(1-q^h)^2} = \prod_{k,r\geq 1}\frac{(1-uq^rT^k)^r(1-u^{-1}q^rT^k)}{(1-q^{r-1}T^k)^r(1-q^{r+1}T^k)}$$

Proof of Dehaye–Han using a specialized Macdonald identity for type $\tilde{A}_t$

Macdonald identity in type $\tilde{C}_t$

$$\sum_{\mathbf{m}\in\mathbb{Z}^t}\sum_{\sigma\in S_t}\mathrm{sgn}(\sigma)\prod_{i=1}^{t} x_i^{(2t+2)m_i}T^{2(t+1)\binom{m_i}{2}+(t+1)m_i}\left((x_iT^{m_i})^{\sigma(i)-t-1}-(x_iT^{m_i})^{t+1-\sigma(i)}\right)$$

$$= \Delta_C(\mathbf{x})\,(T;T)_\infty^t\prod_{i=1}^{t}(Tx_i^{\pm 2};T)_\infty\prod_{1\leq i<j\leq t}(Tx_i^{\pm}x_j^{\pm};T)_\infty,$$

where $\Delta_C(\mathbf{x}) := \prod_{1\leq i\leq t} x_i^{-t}(1-x_i^2)\prod_{1\leq i<j\leq t}(x_j-x_i)(1-x_ix_j)$

A key tool: the Littlewood decomposition using Maya diagrams

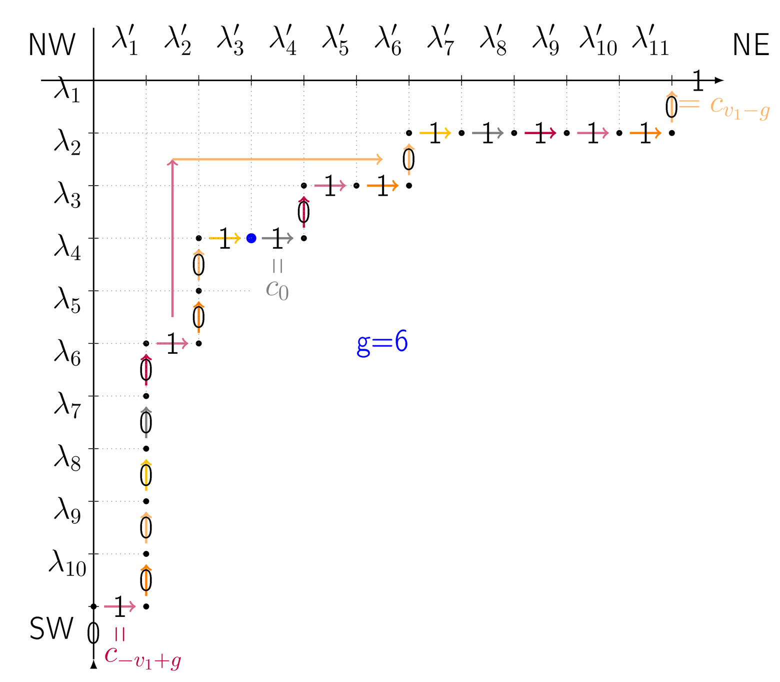
Set $\mathcal{A} \subseteq \mathcal{P}$, $\mathcal{A}_{(t)} := \{\omega \in \mathcal{A} \mid \mathcal{H}_t(\omega) = \emptyset\}$
Littlewood decomposition: bijection such that

$$\lambda \in \mathcal{P} \mapsto (\omega, \underline{\nu}) \in \mathcal{P}_{(t)} \times \mathcal{P}^t$$

Garvan–Kim–Stanton (1990): bijection $\omega \in \mathcal{P}_{(t)} \to \mathbf{n} \in \mathbb{Z}^t$ such that

$$\sum_{i=0}^{t-1} n_i = 0$$

$$|\omega| = \frac{t}{2}\sum_{i=0}^{t-1} n_i^2 + \sum_{i=0}^{t-1} i n_i$$



$V_{g,t}$-coding

t and g positive integers such that $t \leq g$.

- $\lambda \in \mathcal{P} \to s(\lambda) = (c_k)_{k\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$ its corresponding Maya diagram.
- $\beta_i := \max\{(k+1)g + i \mid c_{kg+i} = 0\}$. Let $\sigma : \{1,\dots,g\} \to \{0,\dots,g-1\}$ be the unique bijection such that $\beta_{\sigma(1)} > \cdots > \beta_{\sigma(g)}$.

The vector $\mathbf{v} := (\beta_{\sigma(1)},\dots,\beta_{\sigma(t)})$ is called the $V_{g,t}$-coding corresponding to λ.

Rewriting Macdonald identities in type $\tilde{C}_t$

$$\sum_{\omega\in\mathcal{DD}_{(g)}}(-1)^{d_\omega+|H_+|}T^{|\omega|/2}\mathrm{sp}_\mu(\mathbf{x}) = (T;T)_\infty^t\prod_{i=1}^{t}(Tx_i^{\pm 2};T)_\infty\prod_{1\leq i<j\leq t}(Tx_i^{\pm}x_j^{\pm};T)_\infty,$$

where $g = 2t+2$, $\mathbf{v}$ is the $V_{g,t}$—coding corresponding to ω and $\mu \in \mathcal{P}$ such that $\mu_i := v_i + i - g$ for all $1 \leq i \leq t$.

Hook length enumeration and $V_{g,t}$-coding

Set t a positive integer and $g = 2t+2$. Let $\omega \in \mathcal{DD}_{(g)}$ and $\mathbf{v} \in \mathbb{Z}^t$ its associated $V_{g,t}$-coding, and set $r_i = v_i - t - 1$ for any $i \in \{1,\dots,t\}$. For any function $\tau : \mathbb{Z} \to F^\times$, where F is a field, we have

$$\prod_{s\in\omega}\frac{\tau(h_s-\varepsilon_s g)}{\tau(h_s)} = \prod_{i=1}^{g-1}\left(\frac{\tau(-i)}{\tau(i)}\right)^{\alpha_i(\omega)}\prod_{i=1}^{t}\frac{\tau(r_i)}{\tau(i)}\prod_{1\leq i<j\leq t}\frac{\tau(r_i-r_j)}{\tau(j-i)}\frac{\tau(r_i+r_j)}{\tau(g-i-j)}$$

where $\alpha_i(\omega) := \#\{u \in \omega, h_u = g-i, \varepsilon_u = 1\}$, we have

A q-Nekrasov–Okounkov formula in type $\tilde{C}$

$$\sum_{\lambda\in\mathcal{DD}}(-u)^{d_\lambda}T^{|\lambda|/2}\prod_{s\in\lambda}\frac{1-u^{-2\varepsilon_s}q^{h_s}}{1-q^{h_s}}\prod_{s\in\Delta}\frac{1+uq^{h_s/2}}{1+u^{-1}q^{h_s/2}} = \prod_{m,r\geq 1}\frac{1+uq^{r-1}T^m}{1+u^{-1}q^rT^m}\frac{\left(1-u^{-2}q^{r+2}T^m\right)^{r-\lfloor r/2\rfloor}\left(1-u^{2}q^{r-1}T^m\right)^{r-\lfloor r/2\rfloor}}{\left(1-q^{r}T^m\right)^{r-\lfloor r/2\rfloor}\left(1-q^{r+1}T^m\right)^{r-\lfloor r/2\rfloor}}$$

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