Cyclically symmetric lozenge tilings of a cored hexagon

The equinumerosity extends to horizontally symmetric objects on both sides.

**Arrowed Monotone Triangles**

An arrowed monotone triangle (AMT) is a triangular array of integers,

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 & 5 \\
1 & 2 & 3 & 4 & 5 & 6 \\
2 & 3 & 4 & 5 & 6 & 7 \\
3 & 4 & 5 & 6 & 7 & 8 \\
4 & 5 & 6 & 7 & 8 & 9 \\
5 & 6 & 7 & 8 & 9 & 10 \\
\end{array}
\]

satisfying the above monotonicity conditions together with a decoration of the entries by the symbols "\(\nearrow\), \(\nwarrow\), \(\swarrow\), \(\searrow\)" where an entry can only "point" to entries with a different value.

We define the weight \(W(u,v,w;x)\) of an AMT \(M = (m_{i,j})_{i,j\leq n}\) as

\[
W(u,v,w;x)^n = \prod_{i=1}^{n} \left( u^i + v^i + w^i \prod_{j<i} \left( u^{E_j}_i + v^{E_j}_i + w^{E_j}_i \right) \right) s_k(x_1, x_2, \ldots, x_n),
\]

where the weight of the AMT above is \(W(1,1,1;\cdot,\cdot,\cdot)\).

The generating function of AMTs with bottom row \(k_1 < k_2 < \cdots < k_s\) is

\[
\sum_{k_1 < k_2 < \cdots < k_s} \prod_{j=1}^{s} (u^{x_1+k_{j-1}+1} + v^{x_1+k_{j-1}+1} + w^{x_1+k_{j-1}+1}) s_k(x_1, x_2, \ldots, x_n),
\]

where \(E_j\) is the shift operator defined as \(E_j(x) = x + 1\).

**Extended n-DPP paths**

An extended n-DPP path is a family of lattice paths with starting point \(S_i = (-i, i-2)\) and end points \(E_i = (i, -2)\) for \(i \leq n\) and a given subset \(S \subseteq \{1, \ldots, n\}\) and step set and weights as follows.

The weight of the extended n-DPP path is \(v'(i)\) times the product of the weights of all steps. The weight of the above extended 7-DPP path with \(S = \{1, 2, 4, 6\}\) is

\[
u^1 v^1 w x_1^2 x_2^2 x_3 x_4 x_5 y (1 + w v x_1) (1 + w v x_2) (1 + w v x_3).
\]

**Obtaining the unweighted objects**

When specialising \(u = v = w = x = 1\) and \(w = -1\), we have explicit signed bijections between

1. AMTs with bottom row \((1, 2, \ldots, n)\) and \(n \times n\) ASMs,
2. extended n-DPP paths and DPPs of size \(n\), and in the case of reflective symmetry between
3. AMTs with bottom row \((0, 2, \ldots, 2n-2)\) and \((2n+1) \times (2n+1)\) vertically symmetric ASMs,
4. certain families of lattice paths and cyclically and horizontally symmetric lozenge tilings of a cored hexagon with alternating side lengths \(2n+2\) and \(2n\).

**AMTs and extended n-DPP paths**

The generating function of AMTs with bottom row \((1, 2, \ldots, n)\) and the generating function of extended n-DPP paths are both given by

\[\prod_{j=1}^{n} (x_{j}^{2} + v x_{j} + w x_{j})^{-1} \sum_{T \in \text{TSPP}_{n}} \omega_T(u,v,w) s_{\lambda(T)}(x),\]

where TSPP\(_{n}\) denotes the set of totally symmetric plane partitions inside an \((n, n, n)\)-box, \((a_1, a_2, b_1, \ldots, b_n)\) denotes the Frobenius notation of the diagonal of \(T\), \(\pi_1(T) = (a_1, a_2, b_1 + 1, \ldots, b_n + 1)\), and \(\omega_T(u,v,w) = u^{x_{\lambda(T)}(1)} \cdot \sum_{k=0}^{\lambda(T)} w^{\lambda(T) - k} \sum_{a} \mu_{a}(b-k-a)\).

**Pair of plane partitions with \(n+3\) statistics**

By applying sign-reversing involutions, we obtain pairs \((P, Q)\) of plane partitions of the same shape with \(n\) rows (allowing row length 0) such that

- \(P\) is column-strict and the \(i\)th row from the bottom is bounded by \(2i\);
- \(Q\) is row-strict and the \(i\)th row from the bottom is bounded by \(i\).

The weight is

\[v(q(\lambda)) = \prod_{i=1}^{n} (x_{i}^{2} - q_{i} x_{i} + 1) \in P(v x_{i})^{q(\lambda)} \in P.\]

**Lattice path model for the vertically symmetric case**

The generating function of AMTs with bottom row \((0, 2, \ldots, 2n-2)\) is equal to the generating function of families of lattice paths starting at \(S_i = (-i, i)\), ending at \(E_i = (j-1, -j+2)\), for \(1 \leq i, j \leq n\), and with step set, edge weight and sign given as follows, where \(d\) is the distance from the line \(y = 2\).

The family of paths above has weight \(uvw x_{1}^{2} x_{2}^{2} x_{3} x_{4} x_{5} s_{\lambda}(x_{1} + x_{2} x_{3} - 1)(x_{5} + x_{4} x_{3} - 1)\).

**Expansion into symmetric functions**

The generating function of AMTs with bottom row \((0, 2, \ldots, 2n-2)\) is

\[\prod_{j=1}^{n} (x_{j}^{2} + v x_{j} + w x_{j})^{-1} \sum_{Q} w^{q(\lambda)}(x_{1}, \ldots, x_{n}, x_{1}^{-1}, \ldots, x_{n}^{-1}),\]

where \(\lambda\) denotes the shape of \(Q\) and \(sp\) denotes the generating function of symplectic tableaux.