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Alternating Sign Matrices and Descending Plane Partitions: a linear number of equivalent statistics

Ilse Fischer[a] Hans Höngesberg[b] Florian Schreier-Aigner[c]

a) ilse.fischer@univie.ac.at b) hans.hoengesberg@univie.ac.at c) florian.schreier-aigner@univie.ac.at

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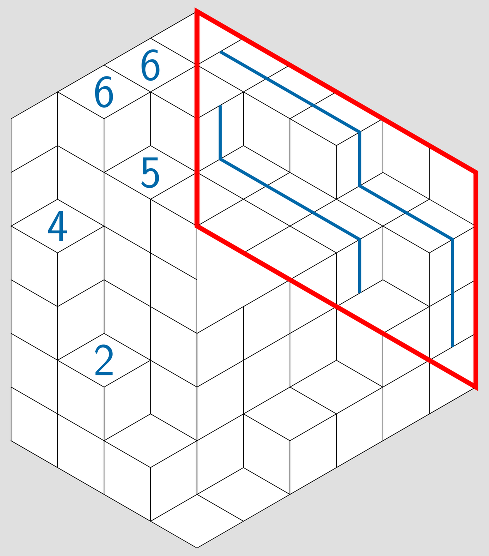
Alternating sign matrices (ASMs) and descending plane partitions (DPPs)

$n \times n$ ASMs — cyclically symmetric lozenge tilings of a cored hexagon with sides $n+1, n-1$

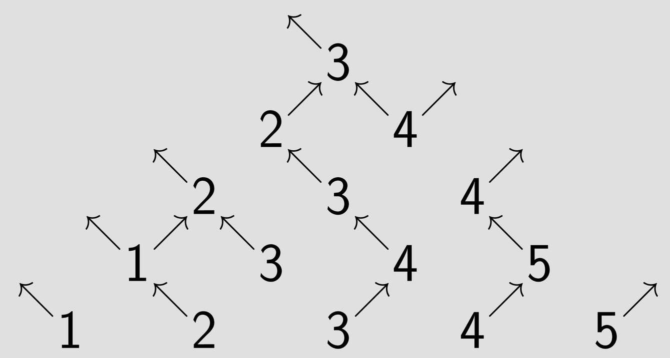
$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad \overset{\text{equinumerous}}{\longleftrightarrow}$$



The equinumerousity extends to horizontally symmetric objects on both sides.

Arrowed Monotone Triangles

An arrowed monotone triangle (AMT) is a triangular array of integers



satisfying the above monotonicity conditions together with a decoration of the entries by the symbols ↖, ↗, ↖↗ where an entry can only "point" to entries with a different value.

We define the weight $W_M(u, v, w; \mathbf{x})$ of an AMT $M = (m_{i,j})_{1 \le j \le i \le n}$ as

$$u^{\#\nearrow} v^{\#\nwarrow} w^{\#\nwarrow\!\!\nearrow} \prod_{i=1}^{n} x_i^{\sum_{j=1}^{i} m_{i,j} - \sum_{j=1}^{i-1} m_{i-1,j} + \#(\nearrow \text{ in row } i) - \#(\nwarrow \text{ in row } i)}.$$

The weight of the AMT above is $u^5 v^8 w^2 x_1^2 x_2^4 x_3^2 x_4 x_5^3$.

Operator Formula

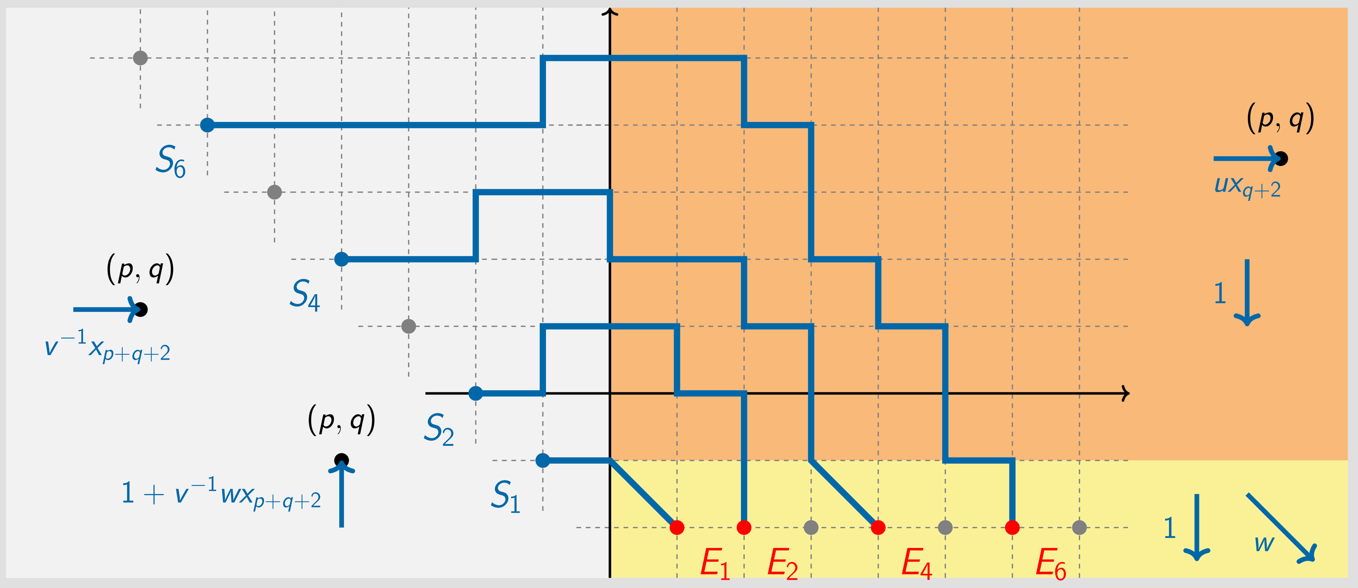
The generating function of AMTs with bottom row $k_1 < k_2 < \ldots < k_n$ is

$$\prod_{i=1}^{n} (u x_i + v x_i^{-1} + w) \times \prod_{1 \le i < j \le n} \left(u \mathrm{E}_{k_i} + v \mathrm{E}_{k_j}^{-1} + w \mathrm{E}_{k_i} \mathrm{E}_{k_j}^{-1} \right) s_{(k_n, k_{n-1}, \ldots, k_1)}(x_1, \ldots, x_n),$$

where E_x denotes the shift operator, defined as $\mathrm{E}_x p(x) = p(x+1)$.

Extended n-DPP paths

An extended n-DPP path is a family of lattice paths with starting points $S_i = (-i, i-2)$ and end points $E_i = (i, -2)$ for $i \in S$ and a given subset $S \subseteq \{1, \ldots, n\}$ and step set and weights as follows.



The weight of an extended n-DPP path is $v^{\binom{n+1}{2}}$ times the product of the weights of all steps. The weight of the above extended 7-DPP path with $S = \{1, 2, 4, 6\}$ is

$$u^{11} v^{15} w^2 x_1^5 x_2^3 x_3^5 x_4^5 x_5^2 x_6 x_7^3 (1 + wv^{-1} x_2)(1 + wv^{-1} x_3)(1 + wv^{-1} x_6).$$

Obtaining the unweighted objects

When specialising $u = v = x_1 = \cdots = x_n = 1$ and $w = -1$, we have explicit signed bijections between

1. AMTs with bottom row $(1, 2, \ldots, n)$ and $n \times n$ ASMs,
2. extended n-DPP paths and DPPs of size n,

and in the case of reflective symmetry between

3. AMTs with bottom row $(0, 2, \ldots, 2n-2)$ and $(2n+1) \times (2n+1)$ vertically symmetric ASMs,
4. certain families of lattice paths and cyclically and horizontally symmetric lozenge tilings of a cored hexagon with alternating side lengths $2n+2$ and $2n$.

AMTs and extended n-DPP paths

The generating function of AMTs with bottom row $(1, 2, \ldots, n)$ and the generating function of extended n-DPP paths are both given by

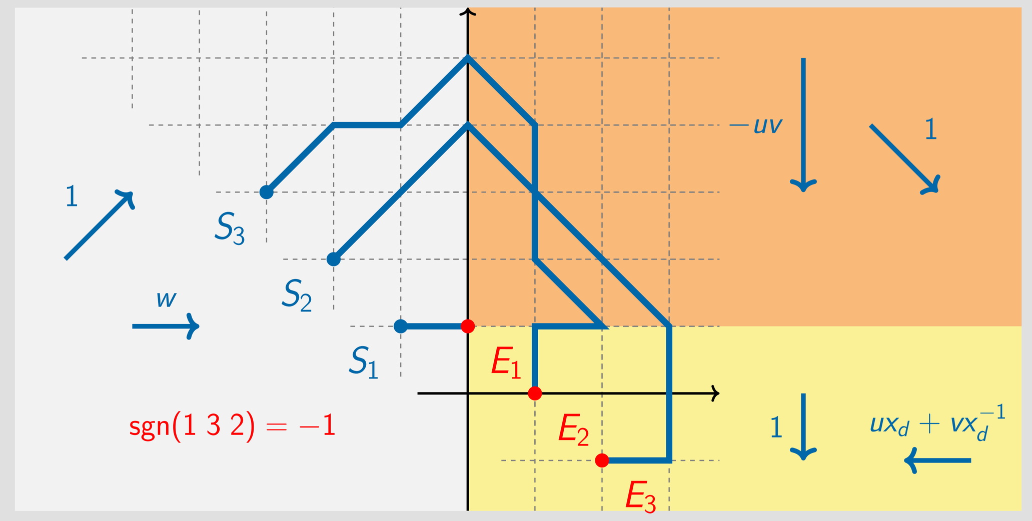
$$\prod_{i=1}^{n} (u x_i^2 + v + w x_i)^{-1} \sum_{T \in \mathrm{TSPP}_{n-1}} \omega_T(u, v, w) s_{\pi_1(T)}(\mathbf{x}),$$

where TSPP_n denotes the set of totally symmetric plane partitions inside an (n, n, n)-box, $(a_1, \ldots, a_l | b_1, \ldots, b_l)$ denotes the Frobenius notation of the diagonal of T, $\pi_1(T) = (a_1, \ldots, a_l | b_1 + 1, \ldots, b_l + 1)$, and

$$\omega_T(u, v, w) = u^{\sum_i (a_i + 1)} v^{\binom{n}{2} - \sum_i (b_i + 1)} w^{\sum_i (b_i - a_i)}.$$

Lattice path model for the verically symmetric case

The generating function of AMTs with bottom row $(0, 2, \ldots, 2n-2)$ is equal to the generating function of families of lattice paths starting at $S_i = (-i, i)$, ending at $E_j = (j-1, -j+2)$, for $1 \le i, j \le n$, and with step set, edge weight and sign given as follows, where d is the distance from the line $y = 2$.



The family of paths above has weight $uvw^2(ux_1 + vx_1^{-1})(ux_3 + vx_3^{-1})$.

Pair of plane partitions with n + 3 statistics

By applying sign-reversing involutions, we obtain pairs (P, Q) of plane partitions of the same shape with n rows (allowing row length 0) such that

- P is column-strict and the ith row from the bottom is bounded by $2i$;
- Q is row-strict and the ith row from the bottom is bounded by i.

The weight is

$$w^{\binom{n+1}{2} - \#\text{ entries in } Q} \prod_{i=1}^{n} x_i^{n-1} (u x_i)^{\#2i-1 \text{ in } P} (v x_i^{-1})^{\#2i \text{ in } P}.$$

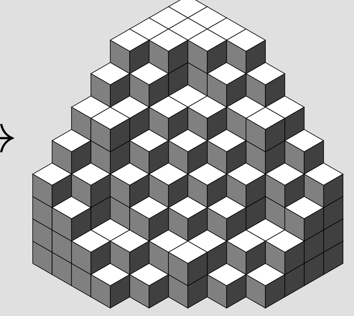
(P, Q) with three rows

Symplectic tableau

$\bar{1}$ 1 / 2 2 / $\bar{3}$ ⟷ 6 ≥ (6 5), 4 ≥ (3 3), 2 ≥ (2)

Totally symmetric self-complementary plane partition

3 ≥ (3 1), 2 ≥ (2 1), 1 ≥ (1) ⟷



Expansion into symmetric functions

The generating function of AMTs with bottom row $(0, 2, \ldots, 2n-2)$ is

$$\prod_i x_i^{n-1} \sum_Q w^{\binom{n+1}{2} - |\lambda(Q)|} \mathrm{sp}_{\lambda(Q)}(ux_1, \ldots, ux_n, vx_1^{-1}, \ldots, vx_n^{-1}),$$

where we sum over all plane partitions Q as above and $\lambda(Q)$ denotes the shape of Q and sp denotes the generating function of symplectic tableaux.