Birational rowmotion over noncommutative rings

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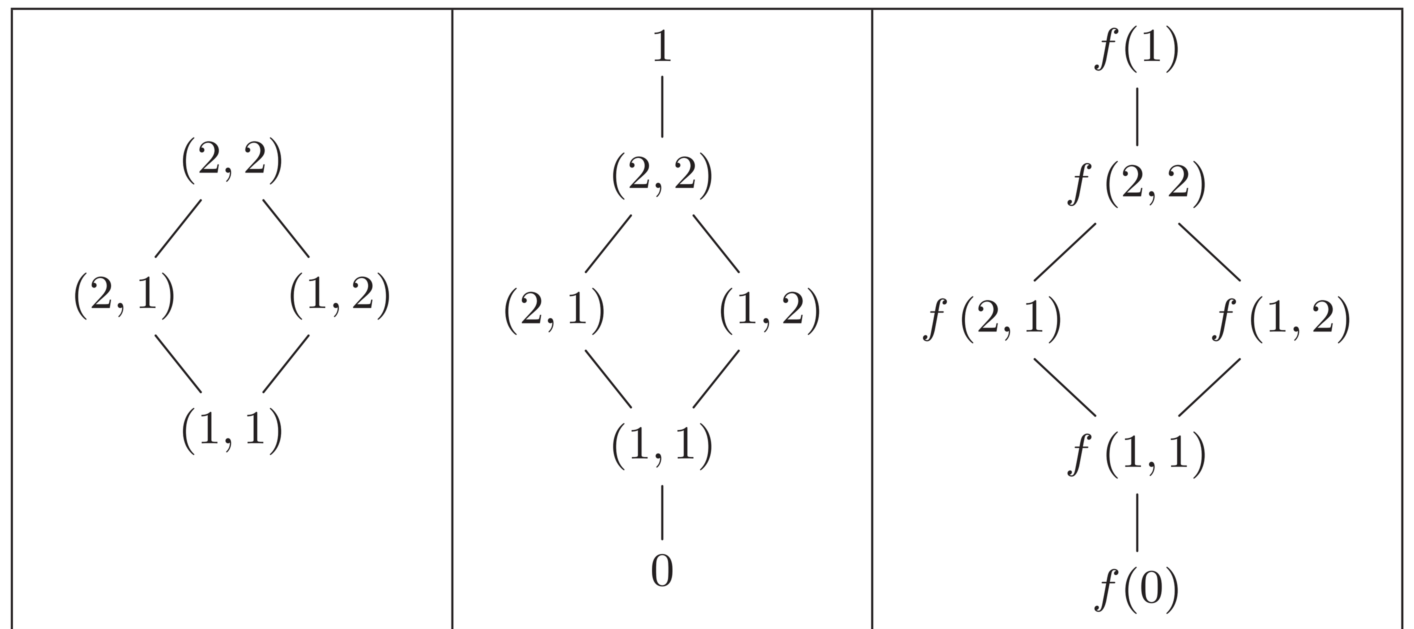
1. Manifest

- **Birational rowmotion** is an birational map defined for every finite poset ([3]).
- It generalizes **PL-rowmotion**, which in turn generalizes **combinatorial rowmotion** but also (quite indirectly) Schützenberger promotion of rectangular SSYTs.
- For various classes of posets, wondrous properties of all types of rowmotion have been surfacing over the last decade ([4], [5], ...).
- Recently, it has been extended to noncommutative rings. Surprisingly, some (not all!) of its good behavior survives!
- We prove that its **periodicity** and **reciprocity** properties survive in the noncommutative case (with a slight twist). See [1] and [2] for details.

2. Notations

- Fix a finite poset P and a ring $\mathbb{K}$ (not necessarily commutative).
- Unless said otherwise, P is a $p \times q$**-rectangle** – i.e., a product $[p] \times [q]$ of two chains $[p] = \{1, 2, \ldots, p\}$ and $[q] = \{1, 2, \ldots, q\}$.
- The notation $\overline{x}$ is short for x^{-1}.
- We let $\widehat{P}$ denote the poset P with two new elements 0 and 1 adjoined to it, with $0 \leq p \leq 1$ for all $p \in \widehat{P}$.
- A $\mathbb{K}$**-labelling** (or, short, **labelling**) of P means a map $f : \widehat{P} \to \mathbb{K}$. Its values $f(p)$ are called its **labels** at the "points" $p \in \widehat{P}$, and we draw it by overlaying these labels at the respective points on the Hasse diagram of $\widehat{P}$. For example:



poset $P = [2] \times [2]$	extended poset $\widehat{P}$	a labelling of P

- The notation "$u \lessdot v$" means "$u < v$, and there is nothing between u and v". Likewise for "$u \gtrdot v$".

8. References

[1] D. Grinberg and T. Roby, *Birational rowmotion over noncommutative rings*, FPSAC 2023 abstract.

[2] D. Grinberg and T. Roby, *Birational rowmotion on a rectangle over a noncommutative ring*, arXiv:2208.11156v3.

[3] D. Einstein and J. Propp, *Combinatorial, piecewise-linear, and birational homomesy for products of two chains*, Algebr. Comb. **4** (2021), pp. 201–224.

[4] S. Hopkins, *Order polynomial product formulas and poset dynamics*, arXiv:2006.01568v5.

[5] T. Roby, *Dynamical algebraic combinatorics and the homomesy phenomenon*, 2016, `https://tinyurl.com/roby-dac`.

3. Noncommutative birational rowmotion

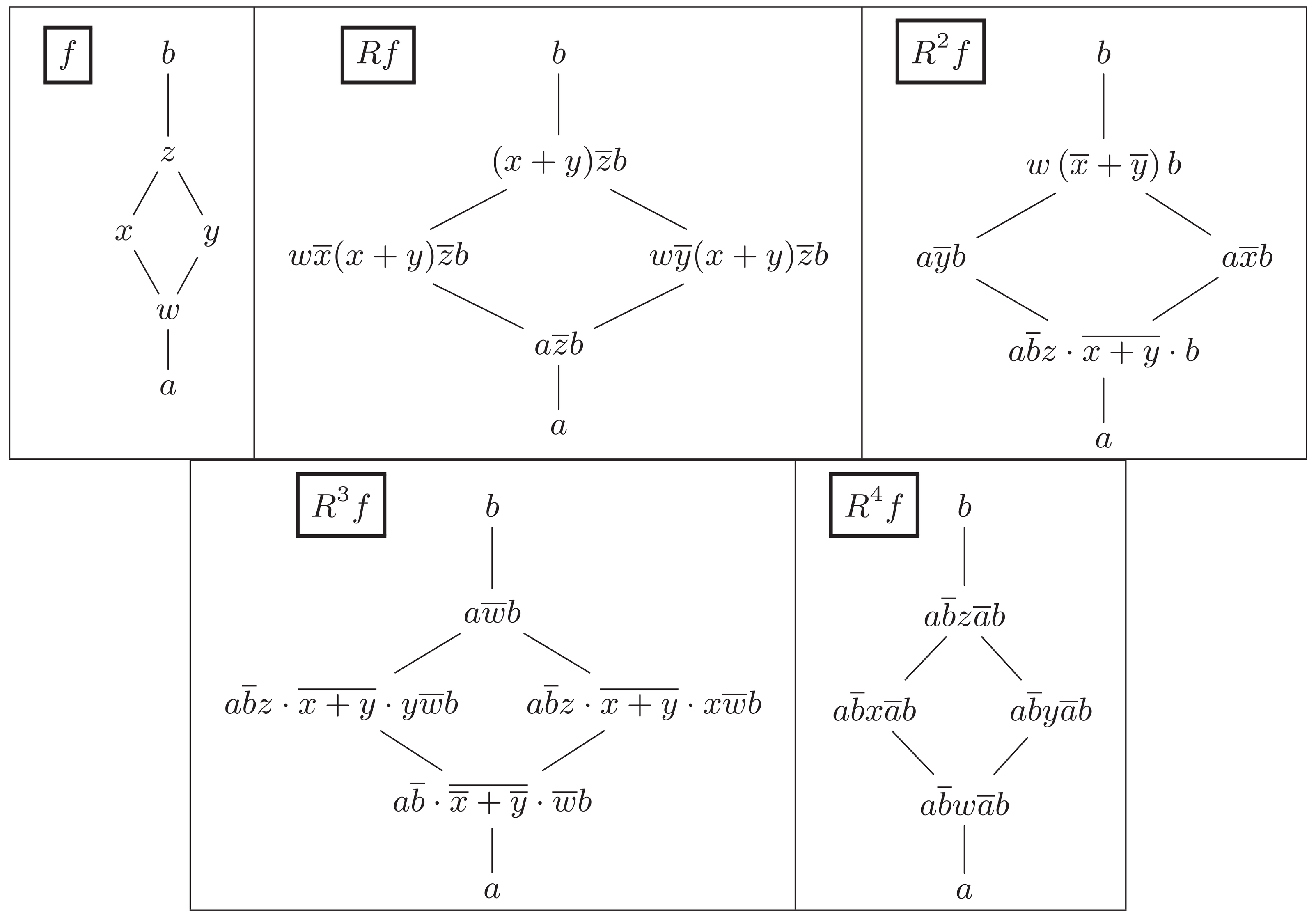
Definition 1. **Birational rowmotion** is the partial map $R : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$ that transforms each labelling f of P into a new labelling Rf whose values are

$$(Rf)(0) = f(0), \qquad (Rf)(1) = f(1), \qquad \text{and}$$

$$(Rf)(v) = \left(\sum_{\substack{u \in \widehat{P};\\ u \lessdot v}} f(u) \right) \cdot \overline{f(v)} \cdot \overline{\sum_{\substack{u \in \widehat{P};\\ u \gtrdot v}} \overline{(Rf)(u)}} \qquad \text{for all } v \in P.$$

- This is a recursive definition: To compute $(Rf)(v)$, the $(Rf)(u)$ for all $u \gtrdot v$ must be known.
- If any of the inverses don't exist, then we leave Rf undefined (thus partial map!).
- Alternatively, R can be defined as a composition of **toggles** (each changing only one label) from top to bottom.
- "Classical" birational rowmotion is obtained when $\mathbb{K}$ is commutative.
- **PL-rowmotion** is the case when $\mathbb{K}$ is the tropical semifield.
- **Combinatorial rowmotion** is the case when $\mathbb{K}$ is the Boolean semiring.

Example 1. Here are the first four iterations of R for $P = [2] \times [2]$:



Note that nontrivial computations with noncommutative inverses are used here!

7. Semirings

- The definition of R and the above theorems are **subtraction-free**, so we can let $\mathbb{K}$ be any semiring.
- However, the proofs use subtraction (in Step 6). Thus another **open problem:** Do periodicity and reciprocity (for $P = [p] \times [q]$) hold when $\mathbb{K}$ is just a semiring?
- This is not obvious! There are several identities (e.g., $a \cdot \overline{a+b} \cdot b = b \cdot \overline{a+b} \cdot a$ when $a + b$ is invertible) that hold in all rings but not in all semirings.

4. The main theorems for rectangles

Let now $P = [p] \times [q]$ for two integers $p, q \geq 1$. Fix a $\mathbb{K}$-labelling $f \in \mathbb{K}^{\widehat{P}}$. Set $a = f(0)$ and $b = f(1)$. Then:

Theorem 1 (twisted periodicity)**.** For any $x \in \widehat{P}$, we have

$$\left(R^{p+q} f\right)(x) = a\overline{b} \cdot f(x) \cdot \overline{a}b \qquad \text{(if } R^{p+q} f \text{ is well-defined).}$$

Theorem 2 (twisted reciprocity)**.** Let $\ell \in \mathbb{N}$ and $(i, j) \in P$ satisfy $\ell - i - j + 1 \geq 0$. Then,

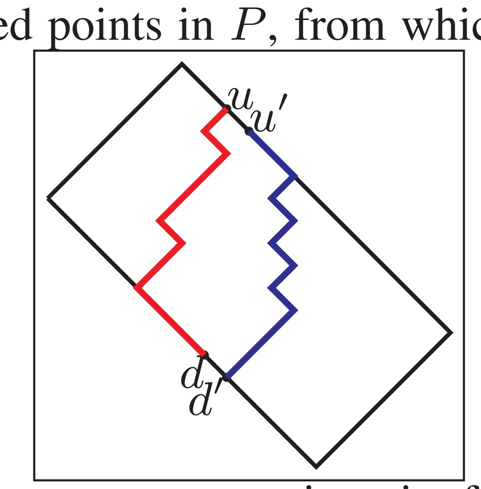
$$\left(R^\ell f\right)(i, j) = a \cdot \overline{\left(R^{\ell-i-j+1} f\right) \underbrace{(p + 1 - i,\ q + 1 - j)}_{\substack{\text{reflection of } (i,j) \text{ through}\\ \text{the center of the rectangle}}}} \cdot b$$

(if $R^\ell f$ is well-defined).

- Note that twisted periodicity just says $R^{p+q}(f) = f$ when $\mathbb{K}$ is commutative.
- Several proofs were known for commutative $\mathbb{K}$, but none of them generalize.
- The commutative case is crypto-equivalent to **Zamolodchikov periodicity** (see [5]), but the latter does not generalize to noncommutative $\mathbb{K}$.

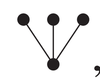
5. Outline of Proof

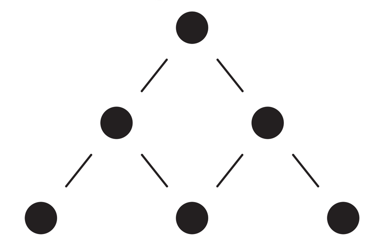
- Suffices to prove Thm 2 from which Thm 1 follows easily in two applications.
- Simplify notation by making a "time subscript" and "antipode tilde" so Thm 2 becomes: $x_\ell = a \cdot \overline{\widetilde{x}_{\ell-i-j+1}} \cdot b$
- Create two types of "slack products," A and $\forall$ to write in condensed form certain expressions along all (lattice) paths between two fixed points in P, from which we can recover the labels u_ℓ for any $u \in P$.
- Use a "conversion lemma" to relate these products when the starting points are adjacent on the upper boundary and ending points are adjacent on the lower boundary. From this, the theorem follows easily for all boundary elements: $A_\ell^{u \to d} = \forall_\ell^{u' \to d'}$.



- Now it is relatively straightforward to induct up the poset to prove reciprocity for interior elements.

6. Other posets

- If $\mathbb{K}$ is commutative, many more classes of posets P are known for which R has nice properties (e.g., periodicity). Some of these persist for general $\mathbb{K}$.
- For $P =$ , periodicity holds for commutative $\mathbb{K}$ but not in general.
- For $P = \Delta(p)$ (type-A positive root poset), periodicity is known to hold for commutative $\mathbb{K}$ and seems to hold in general (as well as reciprocity: $(R^p f)(x) = a\overline{b} \cdot f(x') \cdot \overline{a}b$), but this is an **open problem**.



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