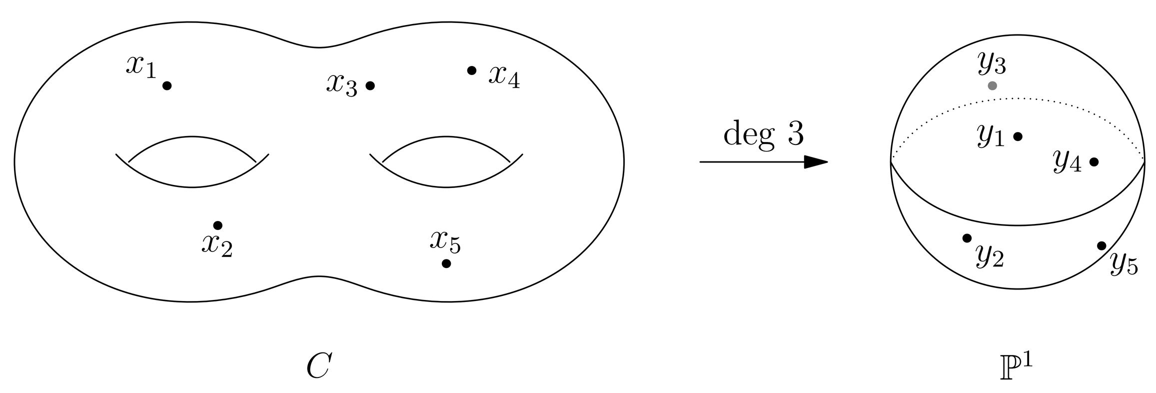
A Generalized RSK for Enumerating Linear Series on n-pointed Curves

Maria Gillespie[1] and Andrew Reimer-Berg[1*]

[1]Colorado State University

Geometric Motivation

Let C be a general genus g curve over $\mathbb{C}$, and let $x_1, \dots, x_n$ be distinct marked points on C. Let $\mathbb{P}^r$ be projective space over $\mathbb{C}$ in r dimensions, and let $y_1, \dots, y_n$ be distinct points in $\mathbb{P}^r$. Consider the degree d morphisms $C \to \mathbb{P}^r$ that send $x_i \to y_i$ for all i:



Let $L_{g,r,d}$ denote the number of such maps. It is known that this number is finite and nonzero if and only if $n = \frac{dr+d+r-rg}{r}$. We make this assumption throughout.

Farkas and Lian [1] showed that for $d \geqslant rg + r$, we have

$$L_{g,r,d} = \int_{\mathrm{Gr}(r+1,d+1)} \sigma^g_{1^r} \cdot \left[\sum_{\alpha_0+\cdots+\alpha_r=(r+1)(d-r)-rg} \left(\prod_{i=0}^{r} \sigma_{\alpha_i} \right) \right] = (r+1)^g$$

They also define a particular structure of Young tableaux that is counted by $L_{g,r,d}$, but do not prove this fact combinatorially.

L-tableaux

Definition. An ***L*-tableau with parameters (g, r, d)** is a way of filling an $(r+1)\times(d-r)$ rectangular grid with:

- **'Red' tableau:** Transposed semistandard Young tableau with r copies of each of the numbers $1, 2, \dots, g$.
- **'Blue' tableau:** Semistandard Young tableau on the remaining skew shape of boxes, with values from $\{0, 1, \dots, r\}$.

Here is an example of an L-tableau with parameters $(4, 3, 9)$:

2	4	1	3	3	3
1	3	4	1	2	2
1	2	3	0	1	1
1	2	3	4	0	0

References

[1] Gavril Farkas and Carl Lian. Linear series on general curves with prescribed incidence conditions. *Journal of the Institute of Mathematics of Jussieu*, pages 1–21, 2022.

[2] Victor Reiner and Mark Shimozono. Percentage-avoiding, northwest shapes and peelable tableaux. *Journal of Combinatorial Theory, Series A*, 82(1):1–73, 1998.

Main Theorem

Theorem 1. *The number of L-tableaux with parameters (g, r, d) is $(r+1)^g$ whenever $d \geqslant g + r$.*

We give a combinatorial proof of this slightly stronger version of Farkas and Lian's result. We prove this by defining an intermediary object (the 'purple' tableaux) that is in bijection with the red tableaux. We then apply the RSK correspondence to the pairs of blue and purple tableaux to form a bijection to $(r+1)$-ary sequences of length g, which are enumerated by $(r+1)^g$.

Proof

Given a red tableau R, we construct a purple tableau $\varphi(R)$ using a specialization of the box complement map used by Reiner and Shimozono [2]:

For each integer $i = 1, 2, \dots, g$, there is a unique row of R **not** containing i. So, starting with $i = 1$, add a box labeled i to $\varphi(R)$ as far to the right as possible in the row that does **not** contain an i in R. Then, iterate for all i.

The tableau $\varphi(R)$ is a rotated standard tableau, and φ is a bijection. The shapes of R and $\varphi(R)$ are complementary in an $(r+1)\times g$ rectangle.

2	4	5	6		
1	3	4	5	7	
1	2	3	5	6	7
1	2	3	4	6	7

7	3	1
	6	2
		4
		5

Given a blue tableau B and a purple tableau $\varphi(R)$ of the same shape, we construct a pair (P, Q) as follows: First, form a semistandard tableau P out of B by rotating 180° and then inverting its content; that is, replace each i with $r - i$ in P. Second, rotate $\varphi(R)$ 180° to form a standard tableau Q.

Via RSK, these pairs are in bijection with $(r+1)$-ary sequences of length g. These sequences are enumerated by $(r+1)^g$.

0	3	3
	1	2
		1
		0

7	3	1
	6	2
		4
		5

$\longrightarrow$

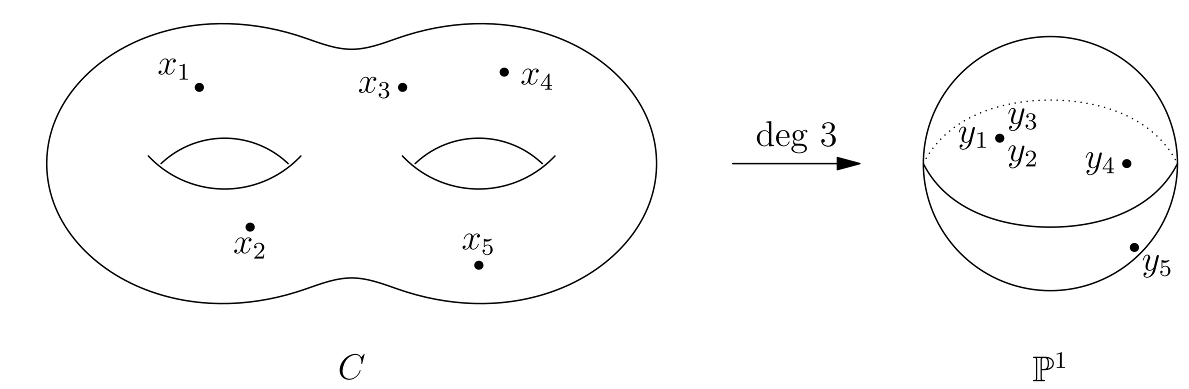
3		
2		
1	2	
0	0	3

5		
4		
2	6	
1	3	7

*andrew.reimer-berg@colostate.edu

Modification: $L'_{g,d,k}$

Restrict to the case where $r = 1$ and we send the first k marked points on C to the same point in $\mathbb{P}^1$. That is, we ask the same question, where now $y_1 = y_2 = \cdots = y_k$. Write $L'_{g,d,k}$ for this new quantity.



Theorem 2. *If $d \geqslant g + k$, we have $L'_{g,d,k} = 2^g$.*

L'-tableaux

Definition. A **positive L'-tableau with parameters (g, d, k)** is a way of filling a $2\times(d-1)$ grid with: a standard Young tableau of size g in the lower left corner (shaded red), a shading of the $k-1$ rightmost boxes in the top row (gray), and a skew semistandard Young tableau in two letters $0, 1$ on the remaining squares (blue).

3	0	1			
1	2	0	0	1	1

We define a **negative L'-tableau with parameters (g, d, k)** analogously, except that it fills a $2\times(d-2)$ grid and has $k-2$ gray boxes.

3	0	1		
1	2	0	0	1

Through the intersection formulas given in [1], $L'_{g,d,k}$ is the difference between the number of positive and negative L' tableaux.

Construct an injection from the negative L'-tableaux to the positive L'-tableaux by adding an extra column with entry 1 in the bottom row.

The difference is precisely the number of positive tableaux with 0 as the final entry. For these tableaux, the entire bottom row (of blue tableau) must be 0, so truncate gray portion. So, this reduces to our earlier case. Since $r = 1$, enumerated by 2^g.

3	0	1			
1	2	0	0	0	0

$\longrightarrow$

3	0	1
1	2	0

$\longrightarrow \Bigg($

3	
1	2

,

1	
0	1

$\Bigg) \longrightarrow 110$

NSF Grant DMS-2054391