A unipotent realization of the chromatic quasisymmetric function

Lucas Gagnon, Department of Mathematics and Statistics, York University

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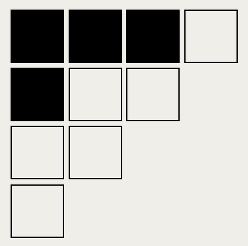
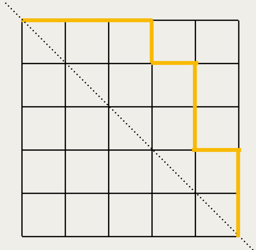
Catalan Families (counted by $C_n = \frac{1}{n+1}\binom{2n}{n}$)

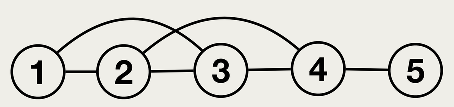
Partition in staircase



Dyck path



Indifference graph



On this poster, "γ"

$\mathbb{F}_q$-Linear Algebraic Groups

Finite field with q elements

On this poster, most matrices are over a finite field with 'q' elements

$$\mathrm{UT}_\gamma = \begin{bmatrix} 1&0&0&*&* \\ 0&1&0&0&* \\ 0&0&1&0&* \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{bmatrix} \subseteq \mathrm{UT}_n = \begin{bmatrix} 1&*&*&*&* \\ 0&1&*&*&* \\ 0&0&1&*&* \\ 0&0&0&1&* \\ 0&0&0&0&1 \end{bmatrix} \subseteq \mathrm{GL}_n$$

"Normal pattern subgroup" (indexed by indifference graph "γ")

Unipotent upper triangular group

General linear group

Guay-Paquet's Hopf Algebra

$\mathrm{IG}(t) = \mathbb{C}[t]$-span{indifference graphs}

On this poster, 't' is always an indeterminate

Set t = 1/q here. (q is a number)

$\mathrm{IG}(q^{-1})$

$\tilde{\gamma} := q^{|E(\gamma)|}\gamma$

$(q-1)^{\deg}$ times Guay-Paquet's Map

≅

The (Hopf) Algebra of Symmetric Functions

$\mathrm{Sym} = \mathbb{C}$-span$\{m_\lambda \mid$ partitions $\lambda\}$

"monomial basis" → $m_\lambda = \sum_{\substack{i_1, i_2, \ldots, i_\ell \\ \text{distinct}}} x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \cdots x_{i_\ell}^{\lambda_\ell}$

Other important bases of Sym:

e_λ ← "elementary basis"

s_λ ← "Schur basis"

$\tilde{P}_\lambda(\mathbf{x}; q)$ ← "Shifted Hall-Littlewood basis"

The Chromatic Quasisymmetric Function

$$X_\gamma(\mathbf{x}; t) = \sum_{\substack{\text{proper colorings} \\ \kappa \text{ of } \gamma}} t^{\mathrm{asc}_\gamma(\kappa)} x_{\kappa(1)} x_{\kappa(2)} \cdots x_{\kappa(n)}$$

"Ascent statistic"

Take these to be indifference graphs

The Unicellular LLT Polynomial

$$G_\gamma(\mathbf{x}; t) = \sum_{\substack{\text{all colorings} \\ \kappa \text{ of } \gamma}} t^{\mathrm{asc}_\gamma(\kappa)} x_{\kappa(1)} x_{\kappa(2)} \cdots x_{\kappa(n)}$$

$(q-1)^n X_\gamma(\mathbf{x}; q)$ Sym → plethysm $f \mapsto \omega f[\frac{\mathbf{x}}{t-1}]|_{t=q}$ → Sym

$\omega G_\gamma(\mathbf{x}; q)$

Theorem [Aguiar—Bergeron—Sottile]: all Hopf morphisms $\Psi : H \to \mathrm{Sym}$ are uniquely determined by the map $\zeta_\Psi = \mathrm{ps} \circ \Psi$, where

$$\mathrm{ps}_1(x_1) = 1 \quad \text{and} \quad \mathrm{ps}_1(x_i) = 0.$$

Theorem [G.]: This square commutes.
Proof uses A–B–S: both directions have the same zeta map, so they much be the same. A $q^{-1} \leftrightarrow q$ result of Shareshian–Wachs is also used.

Theorem [G.]: This square also commutes.
Proof uses results of Carlsson—Mellit and Green, along with the previous square.

≅ $(\mathfrak{F}_\lambda \mapsto \tilde{P}_\lambda(\mathbf{x}; q))$

≅ $(\chi^\lambda \mapsto s_\lambda)$

Superclass Functions

$\mathrm{scf}(\mathrm{UT}_\bullet)$ $\quad \overline{\chi}^\gamma$

Let $\overline{\chi}^\gamma := \mathrm{Ind}_{\mathrm{UT}_\gamma}^{\mathrm{UT}_n}(\mathbb{1}) \in \mathrm{cf}(\mathrm{UT}_n)$, and

$$\mathrm{scf}(\mathrm{UT}_\bullet) = \mathbb{C}\text{-span}\{\overline{\chi}^\gamma\} \subseteq \bigoplus_{n \geq 0} \mathrm{cf}(\mathrm{UT}_n).$$

this space is known to be 'wild', so it's better to work in a simpler subspace like scf

This space has two known Hopf structures. We use a newer one from [G. 23].

Induction from UT_n to GL_n

Class Functions

a.k.a the Hall (Hopf) algebra

$$\begin{aligned} \mathrm{cf}(G) &= \{\psi : G \to \mathbb{C} \mid \psi(g) = \psi(hgh^{-1}) \text{ for all } g, h \in G\} \\ &= \mathbb{C}\text{-span}\{\text{irreducible complex } G\text{-characters}\} \\ &= \mathbb{C}\text{-span}\{\mathbb{C}G\text{-modules}\}/\langle \cong, \oplus \rangle \end{aligned}$$

Unipotent Support

$\mathrm{Ind}_{\mathrm{UT}_\gamma}^{\mathrm{GL}_n}(\mathbb{1})$ → $\mathrm{proj} \circ \mathrm{Ind}_{\mathrm{UT}_\gamma}^{\mathrm{GL}_n}(\mathbb{1})$

$\mathrm{cf}^{\mathrm{uni}}_{\mathrm{supp}}(\mathrm{GL}_\bullet)$ → project onto unipotent summands → $\mathrm{cf}^{\mathrm{uni}}_{\mathrm{char}}(\mathrm{GL}_\bullet)$

Two equivalent definitions:

(1) $\mathrm{cf}^{\mathrm{uni}}_{\mathrm{supp}}(\mathrm{GL}_\bullet) = \bigoplus_{n \geq 0} \mathrm{Ind}_{\mathrm{UT}_n}^{\mathrm{GL}_n}(\mathrm{cf}(\mathrm{UT}_n)) \subseteq \bigoplus_{n \geq 0} \mathrm{cf}(\mathrm{GL}_n)$

(2) $X \in \mathrm{GL}_n$ is *unipotent* if $X - 1$ is nilpotent, and:

$$\mathrm{cf}^{\mathrm{uni}}_{\mathrm{supp}}(\mathrm{GL}_\bullet) = \bigoplus_{n \geq 0} \{\psi \in \mathrm{cf}(\mathrm{GL}_n) \mid \psi(Y) = 0 \text{ if } Y \text{ isn't unipotent}\}$$

(Implicit GL-Hopf Algebra)

$\subseteq \quad \bigoplus_{n \geq 0} \mathrm{cf}(\mathrm{GL}_n) \quad \supseteq$

Hopf structure defined by Zelevinsky. Both $\mathrm{cf}^{\mathrm{uni}}_{\mathrm{supp}}(\mathrm{GL}_\bullet)$ and $\mathrm{cf}^{\mathrm{uni}}_{\mathrm{char}}(\mathrm{GL}_\bullet)$ are sub-Hopf algebras.

Unipotent Characters

Let $B_n = \{$upper triangular matrices in $\mathrm{GL}_n\}$.

$$\mathrm{Ind}_{B_n}^{\mathrm{GL}_n}(\mathbb{1}) = \sum f^\lambda \chi^\lambda$$

#STY(λ)

Unipotent characters

$$\mathrm{cf}^{\mathrm{uni}}_{\mathrm{char}}(\mathrm{GL}_\bullet) = \mathbb{C}\text{-span}\{\chi^\lambda\} \subseteq \bigoplus_{n \geq 0} \mathrm{cf}(\mathrm{GL}_n)$$

$\chi^{(1^n)} = \mathbb{1} \qquad \chi^{(n)} = \mathrm{St}_n$

What About Hessenberg Varieties?

A *Hessenberg Variety* is:

$$\mathcal{B}_M^X = \{gB_n \in \mathrm{GL}_n/B_n \mid g^{-1}Xg \in M\}$$

Upper triangular subgroup

any matrix

Bn-stable subspace, like:

$$\begin{bmatrix} *&*&*&*&* \\ *&*&*&*&* \\ 0&*&*&*&* \\ 0&*&*&*&* \\ 0&0&0&*&* \end{bmatrix} \text{ or } \begin{bmatrix} 0&0&*&*&* \\ 0&0&0&0&* \\ 0&0&0&0&* \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{bmatrix}$$

"Hessenberg space" $\qquad$ UT_γ - 1

We can define them over any field, like $\mathbb{C}$ or $\mathbb{F}_q$. Over $\mathbb{C}$, we get a very nice variety with previously-known connections to the chromatic quasisymmetric function. Over $\mathbb{F}_q$...

Theorem [G.]:

$$\frac{1}{(q-1)^n q^{|E(\gamma)|}} \mathrm{Ind}_{\mathrm{UT}_\gamma}^{\mathrm{GL}_n}(\mathbb{1})(X) = |\mathcal{B}^{X-1}_{\mathrm{UT}_\gamma - 1}| = \mathrm{Poin}(\mathcal{B}^{X-1}_{\mathrm{UT}_\gamma - 1})(q)$$

Poincaré polynomial

Complexify these

Proof uses results of Precup—Sommers, the first commuting square above, and some very easy calculations.

What About Positivity Conjectures?

Can we find a module that affords η_γ and use it to prove this conjecture?

Conjecture [Stanley–Stembrige, Shareshian–Wachs]: The coefficients $a_\lambda(t)$ in $X_\gamma(\mathbf{x}; t) = \sum a_\lambda(t) e_\lambda$ belong to $\mathbb{N}[t]$.

New Conjecture: There are polynomials $a^\gamma_\lambda(t) \in \mathbb{N}[t]$ such that for each prime power q the character $\eta_\gamma = \sum a^\gamma_\lambda(q) \mathrm{St}_{\lambda_1} \mathrm{St}_{\lambda_2} \cdots \mathrm{St}_{\lambda_\ell}$ satisfies $(q-1)^n \eta_\gamma(u) = \mathrm{Ind}_{\mathrm{UT}_\gamma}^{\mathrm{GL}_n}(\mathbb{1})(u)$ for every unipotent element u.

These are equivalent!

Open Problem: It is known that there exist coefficients $b_\lambda(t) \in \mathbb{N}[t]$ for which $G_\gamma(\mathbf{x}; t) = \sum b_\lambda(t) s_\lambda$. Find an explicit formula for $b_\lambda(t)$.

New Open Problem: It is known that there exist coefficients $b_\lambda(t) \in \mathbb{N}[t]$ for which $\mathrm{proj} \circ \mathrm{Ind}_{\mathrm{UT}_\gamma}^{\mathrm{GL}_n}(\mathbb{1}) = \sum b_\lambda(q) \chi^\lambda$. Find an explicit formula for $b_\lambda(t)$.

lgagnon@yorku.ca