# The Kromatic Symmetric Function

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## Conventions for this Poster

- Information in blue is a definition.
- Information in red is an important idea or result.
- The left column is for background information.
- The center column consists of the main definition and results.
- The right column contain other details of interest and contact information.

## Chromatic Symmetric Function

Given a graph $G$ the **chromatic symmetric function** $X_G$ is

$$X_G(x_1, x_2, \dots) = \sum_{\kappa} \prod_{v \in V(G)} x_{\kappa(v)}$$

- $m$- and $p$-basis expansions are well-understood. In particular, setting

$$\widetilde{m}_\lambda = \prod_{i=1}^{\infty}(\#\text{ parts of size i })! m_\lambda,$$

$[\widetilde{m}_\lambda]X_G$ is the number of partitions of $V(G)$ into stable sets of sizes equal to the parts of $\lambda$.

- Stanley-Stembridge conjecture (1995): If $G$ is the incomparability graph of a $(\mathbf{3}+\mathbf{1})$-free poset (see definitions below), then $X_G$ is $e$-positive.

## Vertex-Weighted Chromatic Symmetric Function

A **vertex-weighted graph** $(G, w)$ consists of a graph $G$ and a function $w : V(G) \to \mathbb{Z}^+$. All weights 1 $\implies$ normal graph. Define

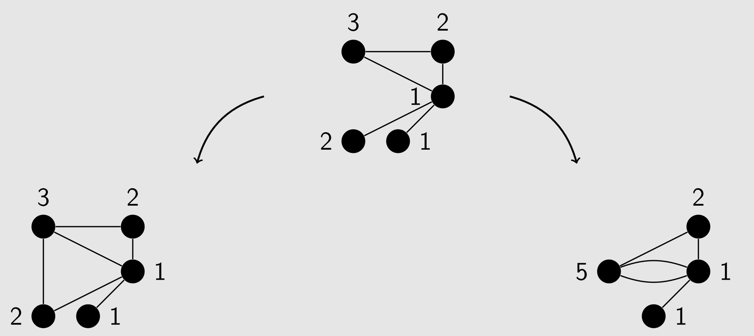
$$X_{(G,w)}(x_1, x_2, \dots) = \sum_{\kappa} \prod_{v \in V(G)} x_{\kappa(v)}^{w(v)}$$

- Admits deletion-contraction relation:

$$X_{(G,w)} = X_{(G \backslash e, w \backslash e)} - X_{(G/e, w/e)}$$

by adding weights of endpoints when contracting edge $e$.

- Gives simple combinatorial method for iterative computation.



Illustrating $X_{(G-e,w)} = X_{(G,w)} + X_{(G/e,w/e)}$

## Motivation: Grothendieck Symmetric Functions

A **multi-valued semistandard Young tableau (MVT)** is a tableau in which each cell contains a *set* of positive integers so that choosing one integer for each cell from its set always gives a semistandard Young tableau.

| | |
|---|---|
| 1, 2 | 2 |
| 4 | 5, 8 |
| 7 | |

The **Grothendieck symmetric function** is

$$\overline{s}_\lambda = \sum_{T \in MVT(\lambda)} (-1)^{\#numbers - \#boxes} \prod_i x_i^{\#i \text{ in } T}.$$

- $K$-theoretic analogues of Schur functions.
- Not homogeneous, lowest degree $|\lambda|$.
- May be viewed as a multiple-valued superposition.

## Other Definitions

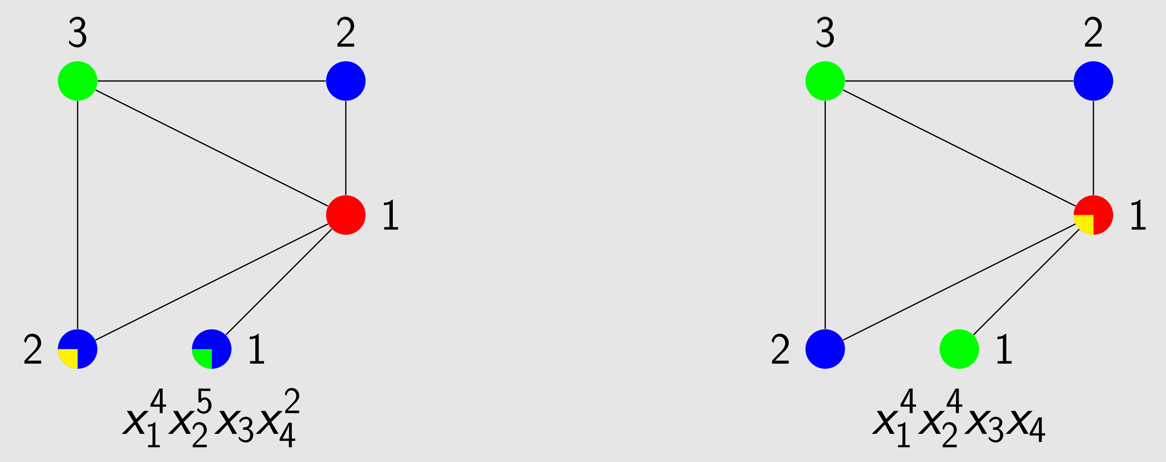
- A **poset** is a set $P$ with a partial order relation $<_P$ that is transitive, nonreflexive, and antisymmetric.
- A poset is **$(3+1)$-free** if it does not contain four elements $a, b, c, d$ such that $a <_P b <_P c$ with $d$ incomparable to all of $a, b, c$.
- The **incomparability graph of a poset** $P$ is a graph with vertex set $P$ and with $v, w \in P$ connected by an edge if and only if $v$ and $w$ are incomparable in $P$.

## Definition and Example

The **Kromatic symmetric function** $\overline{X}_{(G,w)}$ of a vertex-weighted graph is

$$\overline{X}_{(G,w)}(x_1, x_2, \dots) = \sum_{\substack{\kappa: V(G) \to \mathcal{P}(\mathbb{Z}^+) \\ \kappa(v) \neq \emptyset}} \prod_{v \in V(G)} \prod_{i \in \kappa(v)} x_i^{w(v)}$$

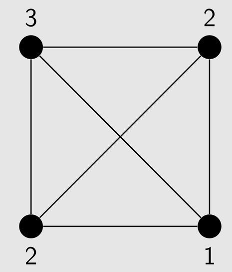
- Lowest-degree terms give $X_{(G,w)}$
- Graph on right: green=1, blue=2, red=3, yellow=4
- Some vertices get two colors




## A Monomial Basis Expansion and a Deletion-Contraction Relation

The expansion of $\overline{X}_{(G,w)}$ in usual symmetric function bases is infinite. It makes sense to find appropriate analogues in the style of $\overline{s}_\lambda$.

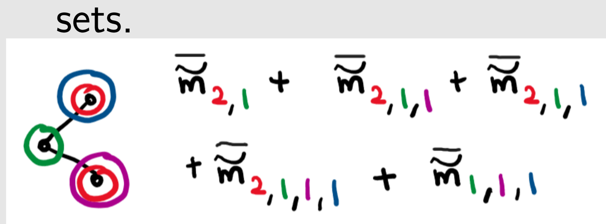
**Define $\overline{\widetilde{m}}_\lambda = \overline{X}_{K^\lambda}$ (below is $K^{(3,2,2,1)}$)**



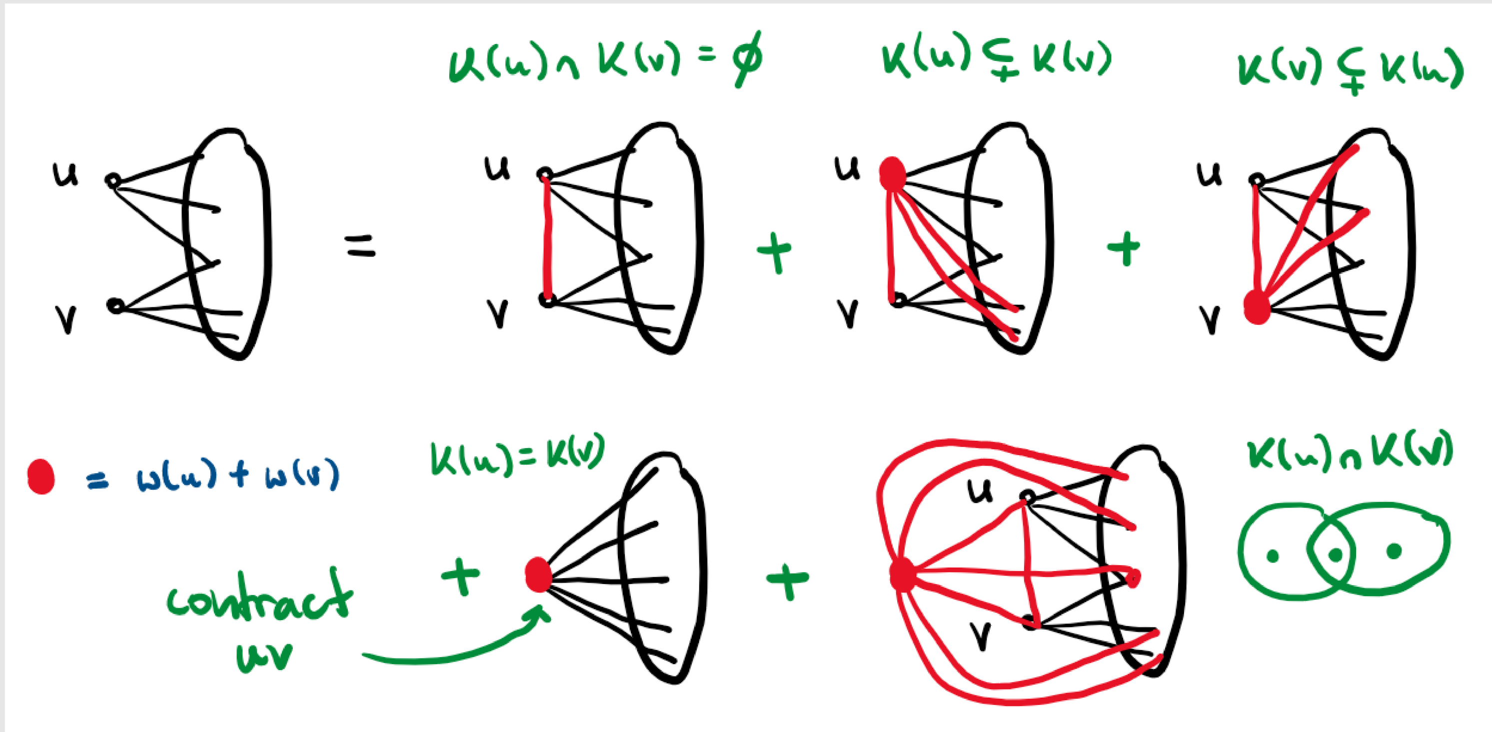

Then

$$\overline{X}_{(G,w)} = \sum_C \overline{\widetilde{m}}_{\lambda(C)}$$

- $C$ covers of $G$ by (not necessarily disjoint) stable sets
- $\lambda(C)$ the weights of the stable sets.



This is an example of the expansion.




- For nonedge $uv$ in $(G, w)$, express $\overline{X}_{(G,w)}$ in terms of $\overline{X}$ of other graphs with $uv$ added or contracted.
- Each graph corresponds to colorings of $(G, w)$ satisfying listed conditions.
- Some other edges are added, e.g. in the upper-right graph, we add edges from $v$ to the neighbors of $u$.
- Repeatedly applying this relation terminates in finite time, giving a method for computing a finite expression for $\overline{X}_{(G,w)}$.

## Expansion in $\overline{s}_\lambda$ Basis

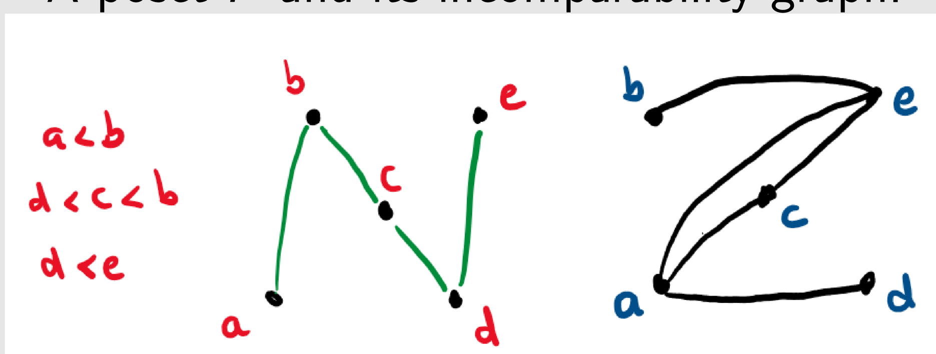
Given a poset $P$, a **$P$-tableau of shape $\mu$** is a filling of $\mu$ with elements of $P$ such that

- Each $p \in P$ occurs at least once.
- Rows are $<_P$-increasing left-to-right.
- Columns are weakly increasing, meaning that if $a$ is immediately above $b$, $a \not>_P b$.

| | | |
|---|---|---|
| d | c | b |
| a | | |
| e | | |

Note that $d$ can be above $e$ in the same column even though $d <_P e$ as long as they are not adjacent.

A poset $P$ and its incomparability graph.




An **elegant $P$-tableau** of shape $\lambda$ is a combination of a $P$-tableau of shape $\mu$ and an elegant tableau of shape $\lambda/\mu$ for some $\mu \subseteq \lambda$.

| | | |
|---|---|---|
| d | c | b |
| a | 1 | 1 |
| e | 2 | |

An **elegant tableau of shape $\lambda/\mu$** is a semistandard Young tableau of this shape so that the entries in row $i$ are no greater than $i-1$.

| | | |
|---|---|---|
| | | |
| | 1 | 1 |
| | 2 | |

## Lifting a Result of Gasharov

**Theorem (Gasharov (1996))**
*When $G$ is the incomparability graph of a $(\mathbf{3}+\mathbf{1})$-free poset $P$, the coefficient of $s_\lambda$ in $X_G$ is the number of $P$-tableaux of shape $\lambda$. In particular, $X_G$ is $s$-positive.*

**Theorem (C.-Pechenik-Spirkl (2023+))**
**When $G$ is the incomparability graph of a $(\mathbf{3}+\mathbf{1})$-free poset $P$, the coefficient of $\overline{s}_\lambda$ in $\overline{X}_G$ is the number of elegant $P$-tableaux of shape $\lambda$. In particular, $\overline{X}_G$ is $\overline{s}$-positive.**

## Proof Sketch for Grothendieck Positivity

Idea: generalize the analogous proof by Gasharov, replacing Schur functions with Grothendieck functions.

- Start with determinantal formula of Lascoux-Naruse (2014) for the dual stable Grothendieck functions which satisfy $\langle \overline{s}_\lambda, \underline{s}_\mu \rangle = \delta_{\lambda\mu}$:

$$\underline{s}_\lambda[x_N] = \det(\underline{s}_{\lambda_i - i + j}[x_N + i - 1])$$

where $f[x_N] = f(x_1, \dots, x_N)$.

- Take enough variables and simplify so all functions have same variable set. Take inner product with $\overline{X}_G$ and reduce to

$$\sum_{\pi \in S_N} sgn(\pi) \sum_{\substack{(l_1, \dots, l_{l(\lambda)}) \\ l_{\pi(i)} \leq \lambda_{\pi(i)} - \pi(i) + i}} \left( \prod_{\pi(i)} \binom{\pi(i) - 1}{l_{\pi(i)}} \right) [m_{\lambda(\pi, l_i)}] \overline{X}_G,$$

where $\lambda(\pi, l_i)$ is the partition with parts $\lambda_{\pi(i)} - \pi(i) + i - l_i$.

- Interpret this as a signed sum over $(\pi, A)$ where $A$ is an **elegant $P$-array** — an elegant $P$-tableau with no restriction on elements in the same column — with row lengths $\lambda_{\pi(i)} - \pi(i) + i$.
- Generalize Gasharov's sign-reversing involution on pairs $(\pi, A)$ whose fixed points are $(id, T)$ where $T$ is an elegant $P$-tableau.
- Key idea: if $A$ is not an elegant $P$-tableau, there is a pair of adjacent cells in a column violating the tableau condition. Find the leftmost, bottommost such pair. Swap portions of rows around this pair as follows:

| | | |
|---|---|---|
| d | c | b |
| e | 1 | 1 |
| a | b | |

$\Longleftrightarrow$

| | | | |
|---|---|---|---|
| d | c | b | |
| e | | | |
| a | b | 1 | 1 |

## $\overline{e}$-positivity?

Question: does the Stanley-Stembridge conjecture lift to $\overline{X}_G$? Can we find an appropriate $\overline{e}$-basis so that this might hold?

Answer: No!

- Two logical options for a multiplicative basis: either $\overline{e}_n = \overline{s}_{1^n}$, or $\overline{e}_n = \frac{1}{n!}\overline{X}_{K_n}$. But for both, even the three-vertex path $P_3$ is not $\overline{e}$-positive!
- $X_{P_3} = 3e_3 + e_{21}$. So for any $\overline{e}$, the degree 3 terms of $\overline{X}_G$ will be $3\overline{e}_3 + \overline{e}_{21}$.
- But both options for the $\overline{e}$-basis give us a monomial $x_1^2 x_2^2$. This does not occur in $\overline{X}_{P_3}$, so must be cancelled by a negative term!

## Further Questions

- Is there a topological interpretation of Schur- and Grothendieck-positivity for $\overline{X}_G$? Is there a subvariety of the Grassmanian that induces both of these functions as structural descriptors?
- Is there a natural $\overline{p}$-basis lifting interpretations on $X_G$? Perhaps one that lifts $\omega(p)$-positivity?
- Does $\overline{X}_G$ distinguish non-isomorphic trees? Perhaps even all non-isomorphic graphs?

## Contact Information

Based on *The Kromatic Symmetric Function: A K-Theoretic Analogue of $X_G$* by

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