Science fiction and Foata-like bijections

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Macdonald polynomials

- $\widetilde{H}_\mu(X;q,t)$ is the *transformed Macdonald polynomial*
- $\{\widetilde{H}_\mu(X;q,t) : \mu \vdash n\}$ is a basis for $\mathbb{Q}(q,t)[x_1,\ldots,x_n]^{Sym}$
- Generalize Schur functions, Hall-Littlewood polynomials, Jack symmetric functions
- Combinatorial formula due to Hagluand, Haiman, and Loehr:

$$\widetilde{H}_\mu(X;q,t) = \sum_{T:\mu\to[n]} q^{inv(T)} t^{maj(T)} x^T$$

where T is a *filling* of μ

Example. $\mu = (3,2)$,

$$T = \begin{array}{|c|c|}\hline 3 & 2 \\ \hline \end{array} \atop \begin{array}{|c|c|c|}\hline 1 & 3 & 1 \\ \hline \end{array}; \quad x^T = x_1^2 x_2 x_3^2, \quad inv(T) = 2, \quad maj(T) = 1$$

- Open problem: find a combinatorial formula for the coefficients in the Schur expansion:

$$\widetilde{H}_\mu(X;q,t) = \sum_{\lambda\vdash n} \widetilde{K}_{\lambda\mu}(q,t) s_\lambda(X)$$

The *Macdonald positivity theorem* asserts that $\widetilde{K}_{\lambda\mu}(q,t) \in \mathbb{N}[q,t]$

Garsia-Haiman modules

- The *Garsia-Haiman module* M_μ is the representation-theoretic analog of $\widetilde{H}_\mu(X;q,t)$
- M_μ is the subspace of $\mathbb{Q}[x_1,\ldots,x_n;y_1,\ldots,y_n]$ spanned by the partial derivatives of a bihomogeneous polynomial $\Delta_\mu = \det(x_i^{p_j} y_i^{q_j})$
- S_n acts by permuting coordinates, endowing M_μ with the structure of a doubly graded S_n-module

Example. $\mu = (2,1)$

$$\begin{array}{|c|}\hline (0,1) \\ \hline \end{array} \atop \begin{array}{|c|c|}\hline (0,0) & (1,0) \\ \hline \end{array} \qquad \Delta_\mu = \det\begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix} = x_2y_3 - x_3y_2 - x_1y_3 + x_3y_1 + x_1y_2 - x_2y_1$$

Six nontrivial partial derivatives:

$$\begin{aligned} \partial_{x_1}(\Delta_\mu) &= -y_3 + y_2 & \partial_{y_1}(\Delta_\mu) &= x_3 - x_2 \\ \partial_{x_2}(\Delta_\mu) &= y_3 - y_1 & \partial_{y_2}(\Delta_\mu) &= -x_3 + x_1 \\ \partial_{x_3}(\Delta_\mu) &= -y_2 + y_1 & \partial_{y_3}(\Delta_\mu) &= x_2 - x_1 \end{aligned}$$

and $\{\Delta_\mu, \partial_{x_1}(\Delta_\mu), \partial_{x_2}(\Delta_\mu), \partial_{y_1}(\Delta_\mu), \partial_{y_2}(\Delta_\mu), 1\}$ is a basis, so $\dim(M_\mu) = 6 = 3!$

- Haiman's $n!$ *theorem* proves that in general $\dim(M_\mu) = |\mu|!$, which in turn proves that $\mathrm{Frob}(M_\mu) = \widetilde{H}_\mu(X;q,t)$
- Kostka-Macdonald coefficients $\widetilde{K}_{\lambda\mu}(q,t)$ describe the doubly graded decomposition of M_μ into irreducible S_n-modules

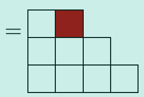
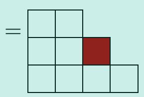
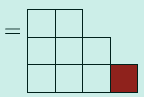
Science fiction

- Bergeron and Garsia studied intersections of k-tuples of Garsia-Haiman modules: the *science fiction heuristics* conjecturally describe the dimension and Frobenius series

Conjecture ($n!/k$ conjecture). *Let $\lambda \vdash n+1$ and let $\mu^1,\ldots,\mu^k \vdash n$ be k distinct partitions obtained by removing a cell from λ. Then*

$$\dim\left(\bigcap_{i=1}^k M_{\mu^i}\right) = \frac{n!}{k}.$$

Example. $\lambda = (4,3,2)$

$\mu^1 =$  $= (4,3,1)$, $\mu^2 =$  $= (4,2,2)$, $\mu^1 =$  $= (3,3,2)$.

Then

$$\dim(M_{(4,3,1)} \cap M_{(4,2,2)} \cap M_{(3,3,2)}) = \frac{8!}{3}.$$

A basis for hook shapes

- Adin, Remmel, and Roichman define a basis for $M_{(a,1^\ell)}$, the Garsia-Haiman module indexed by a hook shape
- Basis elements $\varphi_T \in \mathbb{Q}[x_1,\ldots,x_n;y_1,\ldots,y_n]$ are indexed by *standard fillings* $T : (a,1^\ell) \xrightarrow{\sim} [n]$; x-degree determined by *column inversions* and y-degree determined by *row inversions*

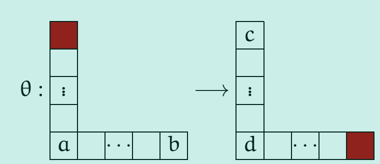
Example.

$$T = \begin{array}{|c|cccc}\hline 4 \\ \cline{1-1} 1 \\ \cline{1-1} 9 \\ \cline{1-1} 7 \\ \hline 5 & \multicolumn{1}{|c|}{6} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{8} \\ \hline \end{array}$$

$$\begin{aligned} colInv(T) &= \{(\mathbf{4},1),(\mathbf{9},7),(\mathbf{9},5),(\mathbf{7},5)\}, \\ rowInv(T) &= \{(5,\mathbf{3}),(5,\mathbf{2}),(6,\mathbf{3}),(6,\mathbf{2}),(3,\mathbf{2})\}, \end{aligned}$$

so $\varphi_T(X,Y) = x_4x_7x_9^2y_2^3y_3^2$.

- We define a bijection θ on certain subsets of standard fillings which preserves appropriate elements in the row/column inversions, determining a basis for $M_{(a,1^{\ell-1})} \cap M_{(a-1,1^\ell)}$



$n!/2$ theorem for hook shapes

The map θ satisfies $\varphi_T = \varphi_{\theta(T)}$ and is defined on a set of size $n!/2$, so θ determines a basis for $M_{(a,1^{\ell-1})} \cap M_{(a-1,1^\ell)}$ and, in particular,

$$\dim(M_{(a,1^{\ell-1})} \cap M_{(a-1,1^\ell)}) = \frac{n!}{2}.$$

Foata's bijection

- Foata defined a family $\{\gamma_x\}$ of bijections on S_n which proves that the statistics maj and inv are equidistributed

Example. To compute $\gamma_5(78136294)$, draw a vertical bar to the right of each number < 5 and cyclically rotate the numbers within each block:

$$7\ 8\ 1|\ 3\ |6\ 2|\ 9\ 4| \quad\longmapsto\quad 1\ 7\ 8|\ 3|\ 2\ 6|\ 4\ 9|$$

So $\gamma_5(78136294) = 17832649$.

- Defining $\Phi(wx) = \gamma_x(\Phi(w))x$, we have $inv(\Phi(w)) = maj(w)$

Foata-like bijections

- The bijection θ is defined by composing two maps on words: arm_u and leg_v

Example. Let $w = 49263187$. To compute $arm_5(w)$ we draw vertical bars to the left of 4, 2, and 3 and within each block shuffle the leftmost number > 5 to the front:

$$|4\ 9\ |2\ 6\ |3\ 1\ 8\ 7 \quad\longmapsto\quad |9\ 4\ |6\ 2\ |8\ 3\ 1\ 7$$

So, $arm_5(w) = 94628317$. The *smaller* entries in the *inversions* are identical in $5w$ and in $arm_5(w)$.

Example. Let $w = 48731926$. To compute $leg_5(w)$ we draw vertical bars to the right of every number 7, 9, and 6 and within each block shuffle the rightmost number < 5 to the end:

$$4\ 8\ 7|\ 3\ 1\ 9|\ 2\ 6| \quad\longmapsto\quad 8\ 7\ 4|\ 3\ 9\ 1|\ 6\ 2|$$

So, $leg_5(48731926) = 87439162$. The *larger* entries in the *coinversions* are identical in $5w$ and in $leg_5(w)$.

Example. Let $\mu = (5,1^4)$ and $\rho = (4,1^5)$, and let

$$T = \begin{array}{|c|cccc}\hline 9 \\ \cline{1-1} 1 \\ \cline{1-1} 7 \\ \cline{1-1} 4 \\ \hline 5 & \multicolumn{1}{|c|}{6} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{8} \\ \hline \end{array} \in SF_<(\mu).$$

$u = 5, v = 6$, so

$$\theta(T) = leg_6 \circ arm_5 \circ bump(T) = leg_6 \circ arm_5 \left(\begin{array}{|c|ccc}\hline 9 \\ \cline{1-1} 1 \\ \cline{1-1} 7 \\ \cline{1-1} 4 \\ \cline{1-1} 5 \\ \hline 6 & \multicolumn{1}{|c|}{\mathbf{3}} & \multicolumn{1}{c|}{\mathbf{2}} & \multicolumn{1}{c|}{\mathbf{8}} \\ \hline \end{array} \right)$$

$$= leg_6 \left(\begin{array}{|c|ccc}\hline \mathbf{9} \\ \cline{1-1} \mathbf{1} \\ \cline{1-1} \mathbf{7} \\ \cline{1-1} \mathbf{4} \\ \cline{1-1} 5 \\ \hline 6 & \multicolumn{1}{|c|}{8} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{2} \\ \hline \end{array} \right) = \begin{array}{|c|ccc}\hline 1 \\ \cline{1-1} 9 \\ \cline{1-1} 4 \\ \cline{1-1} 7 \\ \cline{1-1} 5 \\ \hline 6 & \multicolumn{1}{|c|}{8} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{2} \\ \hline \end{array} \in SF_<(\rho).$$

Note that $\varphi_T = \varphi_{\theta(T)} = x_7^2x_9^4y_2^3y_3^2$.