RSK for 3-free posets

Kyle Celano[1]

[1]Department of Mathematics, University of Miami

Overview

We define an RSK algorithm for chromatic symmetric functions on incomparability graphs of 3-free posets.

Background

The chromatic symmetric function $X_G(\mathbf{x})$ of a graph G was introduced by Stanley as a generalization of the chromatic polynomial. Stanley and Stembridge conjectured e-positivity for incomparability graphs of $\mathbf{3}+\mathbf{1}$-free posets P. For these, Gasharov obtained a combinatorial interpretations of the Schur expansion in terms of P-tableaux through a sign-reversing involution. A bijective proof is unknown except in special cases.

Chromatic symmetric functions

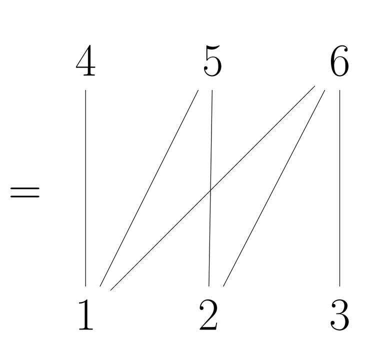
Given a graph $G = (V, E)$, the **chromatic symmetric function** is

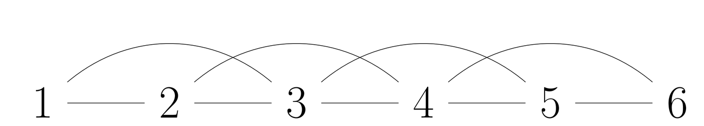
$$X_G(\mathbf{x}) = \sum_{\kappa \in \mathcal{K}(G)} \prod_{v \in V} x_{\kappa(v)}$$

where $\mathcal{K}(G)$ is the set of all proper colorings, which are functions $\kappa : V \to \mathbb{P}$ such that no two adjacent vertices have the same color.

The **incomparability graph** $G = \mathrm{Inc}(P)$ of a poset P has an edge between every pair of incomparable elements of P.

Example

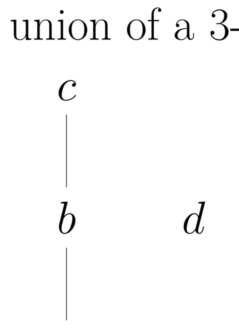
$P =$ 

$G =$ 

$$X_G(\mathbf{x}) = 720m_{1^6} + 144m_{2,1^4} + 28m_{2^2,1^2} + 6m_{2^3}$$

$$X_G(\mathbf{x}) = 6e_{3^2} + 16e_{4,2} + 33e_{5,1} + 96e_6$$

A poset is $\mathbf{3}+\mathbf{1}$-free if it does not contain an induced subposet isomorphic to a disjoint union of a 3-chain and a 1-chain:



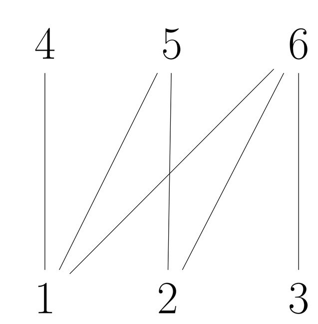
We will be working with posets that are $\mathbf{3}$-free. It is well known that $X_{\mathrm{Inc}(P)}(\mathbf{x})$ is e-positive for $\mathbf{3}$-free posets.

Gasharov's Schur expansion

For a poset P, a **P-tableau** of shape λ is a filling of a Young diagram of shape λ with the elements of P such that

1. the rows are P-nondecreasing, and
2. the columns are strictly P-increasing.

Example



3	1	2	4
6	5		

If $G = \mathrm{Inc}(P)$ for a $\mathbf{3}+\mathbf{1}$-free poset P, then

$$X_G(\mathbf{x}) = \sum_{\lambda \vdash n} |\mathcal{T}_{P,\lambda}| s_{\lambda^*}$$

where $\mathcal{T}_{P,\lambda}$ is set of P-tableaux of shape λ.

Example

$$X_G(\mathbf{x}) = 162s_{1^6} + 66s_{2,1^4} + 22s_{2^2,1^2} + 6s_{2^3}$$

Representing colorings as two-line arrays

To prove Gasharov's expansion bijectively, we want to exhibit an RSK-like map

$$RSK_P : \mathcal{K}(G) \to \bigsqcup_{\lambda \vdash n} \mathcal{T}_{P,\lambda} \times RST_\lambda$$

where RST_λ are **row-strict** tableaux i.e. conjugates of semi-standard Young tableaux.

We first create a **two-line array** out of $\kappa \in \mathcal{K}(G)$ using the **Burge antilexicographic order**.:

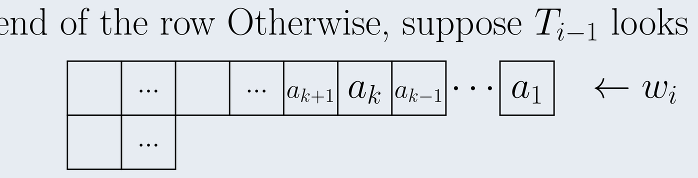
- colors go on top in increasing order,
- corresponding vertices go on bottom,
- order same colored vertices P-decreasingly

Call this two-line array $\binom{\overline{\kappa}}{w}$. Note that $w = w_1 \cdots w_n$ is a permutation of the poset P and that $\overline{\kappa} = \overline{\kappa}_1 \cdots \overline{\kappa}_n$ is a word in $\mathbb{P}$ such that $\overline{\kappa}_i = \kappa(w_i)$.

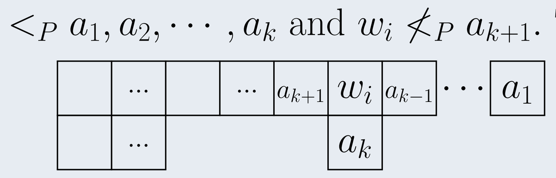
RSK for colorings of 3-free posets

For a coloring $\kappa \in \mathcal{K}(G)$, represent κ as a two-line $\binom{\overline{\kappa}}{w}$. Create a sequence $((T_i, R_i))$ of P-tableaux T_i and row-strict tableaux R_i as follows:

1. $T_1 = \boxed{w_1}$ and $R_1 = \boxed{\overline{\kappa}_1}$
2. Suppose (T_{i-1}, R_{i-1}) has been created. We **insert** w_i into T_{i-1} to create T_i
   1. If $w_i \not<_P$ (last number of the first row of T_{i-1}), then place w_i at the end of the row Otherwise, suppose T_{i-1} looks like



with $w_i <_P a_1, a_2, \cdots, a_k$ and $w_i \not<_P a_{k+1}$. Then make



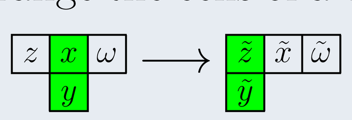
Then, apply the **local moves** moving this new domino to the left until we have a partition shape.

In either case, if c is the new cell of T_i, set $R_i(c) = \overline{\kappa}_i$.

If (T_n, R_n) is the result, then we set $RSK_P(\kappa) = (T_n, R_n)$.

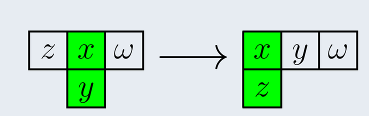
Local moves for 3-free posets

Local moves rearrange the cells of a 4-element subarray

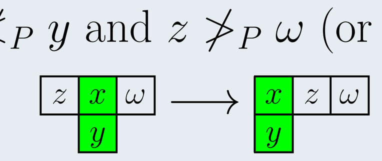


according to the 4 (exclusive) cases

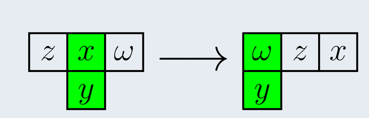
(1) $x <_P z$



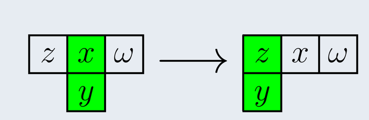
(2a) $x \not<_P z$ and $z \not<_P y$ and $z \not>_P \omega$ (or $\omega = \emptyset$)



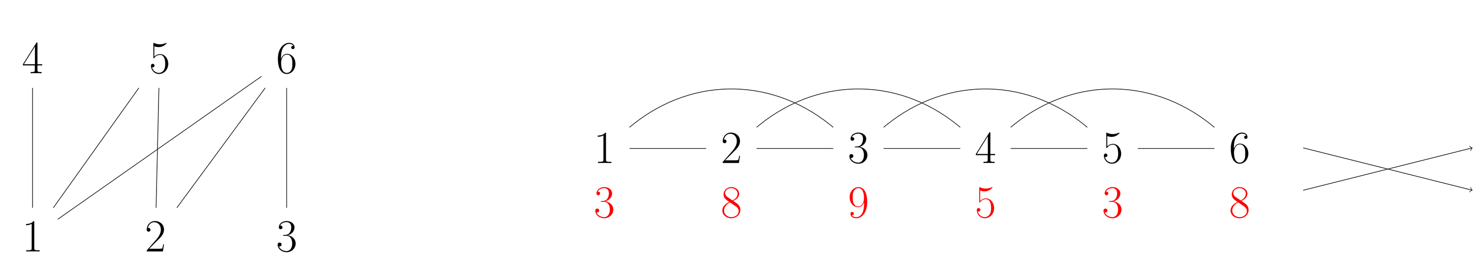
(2b) $x \not<_P z$ and $z \not<_P y$ and $z >_P \omega$



(3) $z <_P y$



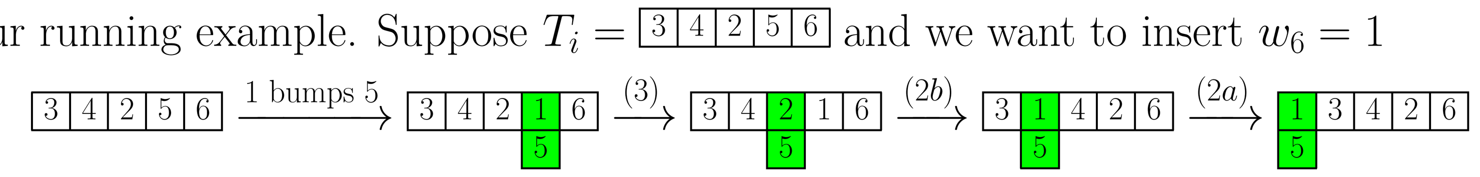
Example (Coloring to two-line array)



$$\begin{pmatrix} 3\,3\,5\,8\,8\,9 \\ 5\,1\,4\,6\,2\,3 \end{pmatrix} = \begin{pmatrix} \overline{\kappa} \\ w \end{pmatrix}$$

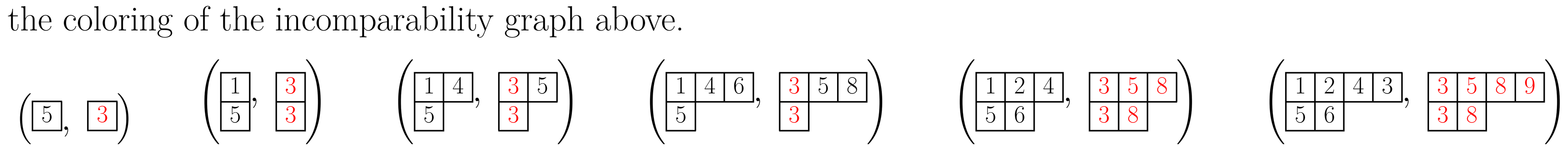
Example (Insertion)

Let P be the poset in our running example. Suppose $T_i = \boxed{3\,4\,2\,5\,6}$ and we want to insert $w_6 = 1$



Example (Full RSK)

Let κ be the coloring of the incomparability graph above.



Theorem (C.)

For a $\mathbf{3}$-free poset P, the map

$$RSK_P : \mathcal{K}(G) \to \bigsqcup_{\lambda \vdash n} \mathcal{T}_{P,\lambda} \times RST_\lambda$$

is a bijection.

For 3-free posets P and $G = \mathrm{Inc}(P)$, RSK_P has nice properties. Try computing them for the Full RSK example.

RSK_P preserves descents

For $w \in \mathfrak{S}_n$ and R a standard Young tableau, define **descent sets**

- $\mathrm{DES}_P(w) = \{i \in [n-1] \mid w_i >_P w_{i+1}\}$
- $\mathrm{DES}(R) = \{i \in [n-1] \mid i \text{ is above } i+1\}$

If κ uses the colors $\{1, \ldots, n\}$ exactly once and $RSK_P(\kappa) = (T, R)$ where $\kappa = \binom{\overline{\kappa}}{w}$, then $\mathrm{DES}_P(w) = \mathrm{DES}(R)$.

Ex: $\mathrm{DES}_P(w) = \{2, 5\}$

RSK_P preserves inversions

Suppose $G = \mathrm{Inc}(P)$ is naturally labeled unit interval order. For a coloring κ and $T \in \mathcal{T}_{P,\lambda}$ define **inversion numbers**

- $\mathrm{inv}_G(\kappa) = \#\{ij \in E \mid i < j, \kappa(i) > \kappa(j)\}$
- $\mathrm{inv}_G(T) = \#\{ij \in E \mid i < j, i \text{ is to the right of } j\}$

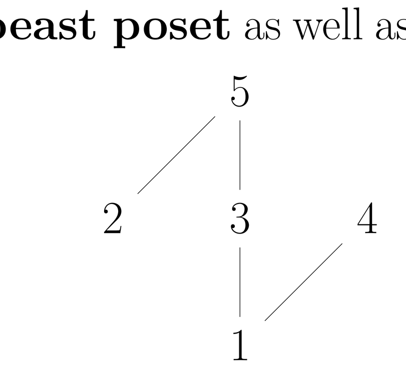
If $RSK_P(\kappa) = (T, R)$, then $\mathrm{inv}_G(\kappa) = \mathrm{inv}_G(T)$.

Ex: $\mathrm{inv}_G(\kappa) = \mathrm{inv}_G(T) = 4$

Further directions

In addition to colorings, there are also **set colorings**, which place sets of colors at vertices so that adjacent vertices are assigned disjoint sets. For $\mathbf{3}$-free posets, RSK_P is a *type* preserving bijection that takes set colorings to pairs (T, R) where T is a **semistandard P-tableau** (can repeat elements).

We would like to be able to extend this algorithm for (set) colorings to all $(\mathbf{3}+\mathbf{1})$-free posets. Currently, we can extend this algorithm to the **beast poset** as well as minor generalization.



For Further Information

- K. Celano, *Chromatic symmetric functions and RSK for (3+1)-free posets*, PhD Dissertation, University of Miami (2023).
- V. Gasharov, *Incomparability graphs of (3 + 1)-free posets are s-positive*, *Disc. Math.* **157** (1996), no. 1-3, 107–125.
- R. P. Stanley, *A symmetric function generalization of the chromatic polynomial of a graph*, *Adv. in Math.* **111** (1995), no. 1, 166–194.