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Wreath Macdonald operators

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ABSTRACT

We construct a novel family of difference-permutation operators and prove that they are diagonalized by the wreath Macdonald P-polynomials. Our operators arise from the action of the horizontal Heisenberg subalgebra in the vertex representation of the quantum toroidal algebra.

1. Wreath Frobenius characteristic

Fix an integer $r > 0$. Consider the *wreath product* of the symmetric group Σ_n and $\mathbb{Z}/r\mathbb{Z}$:

$$\Gamma_n := \Sigma_n \wr \mathbb{Z}/r\mathbb{Z} = \Sigma_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$$

We have a *wreath Frobenius characteristic*:

$$\bigoplus_n \mathrm{Rep}(\Gamma_n) \cong \Lambda^{\otimes r}$$

where Λ is the ring of symmetric functions. The irreducibles $[V_{\vec{\lambda}}] \in \mathrm{Rep}(\Gamma_n)$ are indexed by r-tuples of partitions $\vec{\lambda}$ with $|\vec{\lambda}| = n$. For $\vec{\lambda} = (\lambda^0, \ldots, \lambda^{r-1})$, let

$$s_{\vec{\lambda}} := s_{\lambda^0} \otimes \cdots \otimes s_{\lambda^{r-1}}$$

where $s_\lambda \in \Lambda$ is the Schur function. The wreath Frobenius characteristic sends $[V_{\vec{\lambda}}]$ to $s_{\vec{\lambda}}$.

2. Cores and quotients

For a box $\square = (i,j)$ in a partition, we call $c(\square) := j - i$ its *content*. We call the class of $c(\square)$ mod r its *color*.

The *r-content vector* of a partition λ is the vector

$$(a_0, \ldots, a_{r-1})$$

such that

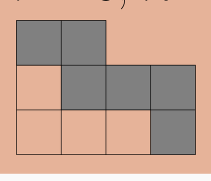
$$a_i = \#\{\square \in \lambda \,|\, c(\square) \equiv i \bmod r\}$$

There is a bijection

$$\{\text{partitions}\} \leftrightarrow \{r\text{-cores}\} \times \{r\text{-tuples of partitions}\}$$
$$\lambda \mapsto (\mathrm{core}(\lambda), \mathrm{quot}(\lambda))$$

Each square in the *r-quotient* $\mathrm{quot}(\lambda)$ records a ribbon of length r in λ—the square sits in the ith coordinate if the northwesternmost square of the ribbon has color i. The *r-core* $\mathrm{core}(\lambda)$ records what is left over when all ribbons of length r are peeled off.

Example: $r = 3$, $\lambda = (4,4,2)$



$$\mapsto \begin{matrix} \mathrm{core}(\lambda) = (3,1) \\ \mathrm{quot}(\lambda) = ((0),(0),(2)) \end{matrix}$$

FACT: $\mathrm{core}(\lambda)$ is determined by the r-content vector mod the diagonal, i.e. by an element of the A_{r-1} root lattice Q.

3. Wreath Macdonald polynomials

Let $\mathfrak{h}_n$ be the reflection representation of Γ_n. Haiman [1] formulated the following definition:

Definition: For λ with $|\mathrm{quot}(\lambda)| = n$, $H_\lambda \in \mathbb{C}(q,t) \otimes \mathrm{Rep}(\Gamma_n)$ is characterized by

1. $H_\lambda \otimes \sum_{i=0}^n (-q)^i \bigwedge^i [\mathfrak{h}_n^*]$ lies in the span of $\{[V_{\mathrm{quot}(\mu)}] \,|\, \mathrm{core}(\mu) = \mathrm{core}(\lambda)$ and $\mu \geq \lambda\}$;
2. $H_\lambda \otimes \sum_{i=0}^n (-t)^{-i} \bigwedge^i [\mathfrak{h}_n^*]$ lies in the span of $\{[V_{\mathrm{quot}(\mu)}] \,|\, \mathrm{core}(\lambda) = \mathrm{core}(\mu)$ and $\mu \leq \lambda\}$;
3. the coefficient of the trivial representation is 1.

This is a generalization of the *transformed* Macdonald polynomials. Any fixed r-core produces an ordering on r-tuples of partitions by using the core-quotient bijection and dominance order on single partitions.

The P-polynomial P_λ is obtained by performing the tensor product in condition (2), inverting t, and then normalizing so that the coefficient of $[V_{\mathrm{quot}(\lambda)}]$ is 1.

4. Finitization

For finitely many variables, we will use an alphabet for each tensorand of $\Lambda^{\otimes r}$, i.e. $\{x_a^{(i)}\}_{a=1}^{N_i}$ for the ith tensorand. Let

$$N_\bullet = (N_0, \ldots, N_{r-1}), \qquad X_{N_\bullet} = \bigcup_{i \in \mathbb{Z}/r\mathbb{Z}} \{x_a^{(i)}\}_{a=1}^{N_i}$$

When specializing P_λ, we impose the following:

Compatibility condition: $N_\bullet$ and the r-content vector of λ are congruent mod the diagonal.

This is really a compatibility between $N_\bullet$ and $\mathrm{core}(\lambda)$.

5. Cyclic-shift operators

Define a *shift pattern of* $X_{N_\bullet}$ to be a subset of $X_{N_\bullet}$ that contains no more than one variable of each color. A shift pattern *contains the color* $p \in \mathbb{Z}/r\mathbb{Z}$ if it contains a variable of color p. Let $Sh_p(X_{N_\bullet})$ denote the set of all shift patterns containing p.

For a shift pattern $\underline{J}$, let $J \subset \mathbb{Z}/r\mathbb{Z}$ denote the set of colors of the variables in $\underline{J}$. We denote the variables in $\underline{J}$ by $x_{\underline{J}}^{(i)}$, so $\underline{J} = \{x_{\underline{J}}^{(i)}\}_{i \in J}$. To $\underline{J}$ we associate the following:

1. *Gap labels:* For $i \in \mathbb{Z}/r\mathbb{Z}$ let $i^\triangledown \in J$ be the first element less than or equal to i in the cyclic order. We stipulate that $0 \leq i - i^\triangledown \leq r-1$. Define
$$x_{\underline{J}^\triangledown}^{(i)} = q^{(i - i^\triangledown)} x_{\underline{J}}^{(i^\triangledown)}.$$
In particular $x_{\underline{J}^\triangledown}^{(i)} = x_{\underline{J}}^{(i)}$ if $i \in J$.
2. *A cyclic-shift operator:* For $i \in J$, let $i^\blacktriangledown \in J$ be the first element *strictly* less than i in the cyclic order. We set $1 \leq i - i^\blacktriangledown \leq r$, where r occurs if and only if $|J| = \{i\}$. We then define the operator $T_{\underline{J}}$ on $\mathbb{C}(q,t)[X_{N_\bullet}]$ by
$$T_{\underline{J}}(x_l^{(i)}) = \begin{cases} q^{(i-i^\blacktriangledown)} x_{\underline{J}}^{(i^\blacktriangledown)} & \text{if } i \in J \text{ and } x_l = x_{\underline{J}}^{(i)} \\ x_l^{(i)} & \text{otherwise.} \end{cases}$$

For an n-tuple $\underline{\mathbf{J}} = (\underline{J}_1, \ldots, \underline{J}_n)$ of shift patterns and $0 \leq k \leq n$, we denote

$$|\underline{\mathbf{J}}| = \underline{J}_1 \cup \cdots \cup \underline{J}_n \subset X_{N_\bullet},$$
$$|\underline{\mathbf{J}}|_{\leq k} = \underline{J}_1 \cup \cdots \cup \underline{J}_k \subset X_{N_\bullet},$$
$$|\underline{\mathbf{J}}|_{\geq k} = \underline{J}_k \cup \cdots \cup \underline{J}_n \subset X_{N_\bullet}.$$

If $\underline{\mathbf{J}}$ is an n-tuple of shift patterns all containing color p, we say $\underline{\mathbf{J}}$ is *p-distinct* if the p-colored variables $x_{\underline{J}_a}^{(p)}$ are all distinct. Let $Sh_p^{[n]}(X_{N_\bullet})$ denote the set of all p-distinct n-tuples of shift patterns containing color p.

6. Wreath Macdonald operators

The wreath Macdonald operators depend on a color $p \in \mathbb{Z}/r\mathbb{Z}$ and degree $n \leq N_p$:

$$D_{p,n}(X_{N_\bullet}; q,t) := \frac{(-1)^{\frac{n(n-1)}{2}}}{\prod_{k=1}^n (1 - q^k t^{-k})} \sum_{\underline{\mathbf{J}} \in Sh_p^{[n]}(X_{N_\bullet})} \overset{\curvearrowright}{\prod_{a=1}^{n}} \left\{ (1 - qt^{-1})^{|\underline{J}_a|} \left(\frac{x_{\underline{J}_a^\triangledown}^{(r-1)}}{x_{\underline{J}_a}^{(p)}} \right) \frac{\prod_{\substack{l=1 \\ x_l^{(p)} \notin |\underline{\mathbf{J}}|_{\geq a}}}^{N_p} \left(t x_{\underline{J}_a^\triangledown}^{(p-1)} - x_l^{(p)} \right)}{\prod_{\substack{l=1 \\ x_l^{(p)} \notin |\underline{\mathbf{J}}|_{\leq a}}}^{N_p} \left(x_{\underline{J}_a}^{(p)} - x_l^{(p)} \right)} \right.$$
$$\left. \times \left(\prod_{\substack{i \in \mathbb{Z}/r\mathbb{Z} \\ i \neq p}} \prod_{\substack{l=1 \\ x_l^{(i)} \neq x_{\underline{J}_a^\triangledown}^{(i)}}}^{N_i} \frac{\left(t x_{\underline{J}_a^\triangledown}^{(i-1)} - x_l^{(i)} \right)}{\left(x_{\underline{J}_a^\triangledown}^{(i)} - x_l^{(i)} \right)} \right) \left(\prod_{i \in J_a \setminus \{p\}} \frac{q^{-1} t T_{\underline{J}_a} x_{\underline{J}_a}^{(i)}}{\left(x_{\underline{J}_a}^{(i)} - T_{\underline{J}_a} x_{\underline{J}_a}^{(i)} \right)} \right) T_{\underline{J}_a} \right\}.$$

7. Main result

Let

$$|N_\bullet| = \sum_{i \in \mathbb{Z}/r\mathbb{Z}} N_i.$$

Our main result [3] is:

Theorem: For λ having r-core compatible with $N_\bullet$ and $\ell(\lambda) \leq |N_\bullet|$, $P_\lambda[X_{N_\bullet}; q,t]$ satisfies (and is uniquely determined by) the eigenfunction equations:

$$D_{p,n}(X_{N_\bullet}; q,t) P_\lambda[X_{N_\bullet}; q,t] = e_n \left[\sum_{\substack{b=1 \\ b - \lambda_b \equiv p+1 \bmod r}}^{|N_\bullet|} q^{\lambda_b} t^{|N_\bullet| - b} \right] P_\lambda[X_{N_\bullet}; q,t].$$

For the eigenvalue, we have used plethystic notation—we merely mean the elementary symmetric polynomial evaluated at the characters within the summation. In the case $r = 1$, we do indeed obtain the usual Macdonald operators after some simplification.

8. Quantum toroidal and shuffle algebras

Our proof relies on work of the third author [4], which relates wreath Macdonald polynomials to the rank r *quantum toroidal algebra* $U_{\mathfrak{q},\mathfrak{d}}(\ddot{\mathfrak{sl}}_r)$ and its *vertex representation* W. Specifically, W can be identified with $\mathbb{C}(q,t) \otimes \Lambda^{\otimes r} \otimes \mathbb{C}[Q]$ as a vector space, and there is a natural way to situate $\{P_\lambda\}$ in W such that they diagonalize the *horizontal Heisenberg subalgebra* of $U_{\mathfrak{q},\mathfrak{d}}(\dot{\mathfrak{sl}}_\ell)$. We discovered our operators by explicitly computing the action on W of well-chosen elements of this subalgebra. To carry out this computation, we use work of Neguț [2] realizing $U_{\mathfrak{q},\mathfrak{d}}(\ddot{\mathfrak{sl}}_\ell)$ as a *shuffle algebra*.

References

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