1. Wreath Frobenius characteristic

For an integer \( r > 0 \), consider the wreath product of the symmetric group \( S_r \) and the integers \( \mathbb{Z}/r\mathbb{Z} \):

\[
G_r = S_r \wr \mathbb{Z}/r\mathbb{Z} = S_r \times (\mathbb{Z}/r\mathbb{Z})^r
\]

We have a wreath Frobenius characteristic:

\[
\bigoplus \text{Rep}(G_r) \cong \Lambda^{\text{rw}}
\]

where \( \Lambda \) is the ring of symmetric functions. The irreducibles \( [\nu] \in \text{Rep}(G_r) \) are indexed by \( r \)-tuples of partitions \( \lambda \) with \( |\lambda| = n \). For \( \lambda = (\lambda_1, \ldots, \lambda_r) \), let

\[
x^{\lambda} = x_{\lambda_1} \cdots x_{\lambda_r}
\]

where \( x_s \) is the Schur function. The wreath Frobenius characteristic sends \( [\nu] \) to \( x^{\nu} \).

2. Cores and quotients

For a box \( \square = (i, j) \) in a partition, we call \( \ell(\square) = j - i \) its content. We call the class of \( \square \) mod its color.

The \( r \)-content vector of a partition \( \lambda \) is the vector

\[
(a_0, \ldots, a_{r-1})
\]

such that

\[
a_i = \#(\square \in \lambda : \ell(\square) \equiv i \mod r)
\]

There is a bijection

\[
\{\text{partitions}\} \leftrightarrow \{\text{cores}\} \times \{\text{\( r \)-tuples of partitions}\}
\]

\( \lambda \mapsto (\text{core}(\lambda), \text{quot}(\lambda)) \)

Each square in the \( r \)-quotient of \( \lambda \) is a ribbon of length \( r \) in \( \lambda \)–the square in the \( i \)-th coordinate if the northwesternmost square of the ribbon has color \( i \). The \( r \)-core \( \text{core}(\lambda) \) records what is left over when all ribbons of length \( r \) are peeled off.

Example: \( r = 3, \lambda = (4, 4, 2) \)

\[
\begin{array}{c}
\text{core}(\lambda) = (3, 3, 1) \\
\text{quot}(\lambda) = ((0, 0), (0, 0), (2, 2))
\end{array}
\]

The \( r \)-content of \( \text{core}(\lambda) \) is determined by \( \lambda \), the \( r \)-content vector mod the diagonal, i.e. by an element of the \( A_{r-1} \) root lattice \( Q \).

3. Wreath Macdonald polynomials

Let \( B_\lambda \) be the reflection representation of \( G_r \). Haiman [1] formulated the following definition:

**Definition:** For \( \lambda \) with \( q\text{quot}(\lambda) = n \), \( B_\lambda \in \text{C}(q,t) \otimes \text{Rep}(G_r) \).

- 1. \( B_\lambda \cong \bigoplus_{\mu \geq \lambda} V_{[\mu]} \otimes [\mu] \)
- 2. \( B_\lambda \cong \bigoplus_{\mu \geq \lambda} \otimes_{\mu} \bigoplus_{\nu \geq \lambda} [\nu] \otimes V_{[\nu]} \)
- 3. The coefficient of the trivial representation is 1.

This is a generalization of the transformed Macdonald polynomials. Any fixed \( r \)-core produces an ordering on \( r \)-tuples of partitions by using the core-quotient bijection and dominance order on single partitions.

The \( P \)-polynomial \( P_r \) is obtained by performing the tensor product in condition (2), inverting \( t \), and then normalizing so that the coefficient of \( V_{[\lambda]} \otimes [\lambda] \) is 1.

4. Finitization

For finitely many variables, we will use an alphabet for each tensor of \( \Lambda^{an} \), i.e. \( x^{(i)}_{\mu} \), for \( i \) the tensor.

\[
N_r = (N_0, \ldots, N_{r-1}), \quad X_r = \bigcup_{n \in \mathbb{Z}/r\mathbb{Z}} (x^{(i)}_{\mu})_{\mu \geq \lambda}
\]

When specializing \( P_r \), we impose the following:

**Compatibility condition:** \( N_r \) and the \( r \)-content vector of \( \lambda \) are congruent mod the diagonal.

This is really a compatibility between \( N_r \) and \( \text{core}(\lambda) \).

5. Cyclic-shift operators

Define a shift pattern of \( X_r \) to be a subset of \( X_r \) that contains no more than one variable of each color. A shift pattern contains the color \( p \in \mathbb{Z}/r\mathbb{Z} \) if it contains a variable of color \( p \). Let \( S_{\lambda}(X_r) \) denote the set of all shift patterns containing \( p \).

- For a shift pattern \( \Delta \), let \( J \subset \mathbb{Z}/r\mathbb{Z} \) denote the set of colors of the variables in \( \Delta \). We denote the variables in \( \Delta \) by \( x^{\Delta} \).

\[
\Delta \mapsto [\nu]_{\lambda} \mapsto \prod_{i \in J} x^{\Delta} \prod_{i \notin J} x^{\lambda_i}
\]

6. Wreath Macdonald operators

The wreath Macdonald operators depend on a color \( p \in \mathbb{Z}/r\mathbb{Z} \) and degree \( n \leq N_r \).

\[
D_{\lambda}(X_r, q, t) = \prod_{\nu \geq \lambda} \prod_{\mu \geq \lambda} \left( 1 - qt \right) \frac{\prod_{\sigma \in \Sigma_{\lambda}} \prod_{i=1}^{\mu_i} (x^{(i)}_{\mu} - x^{(i)}_{\nu})}{\prod_{\sigma \in \Sigma_{\lambda}} \prod_{i=1}^{\mu_i} (x^{(i)}_{\nu} - x^{(i)}_{\mu})}
\]

References


