Combinatorial invariance for elementary intervals using hypercube decompositions

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Abstract

We prove the combinatorial invariance conjecture for *elementary intervals* of the symmetric group. This conjecture posits that Kazhdan–Lusztig polynomials associated to intervals in the Bruhat order depend only on the poset structure of the interval. Our proof uses *hypercube decompositions*, which were recently introduced by Blundell, Buesing, Davies, Veličković, and Williamson. Instead of studying the Kazhdan–Lusztig polynomials directly, we apply hypercube decompositions to the related family of $\widetilde{R}$-polynomials.

Background

Let $W = S_n$ be the symmetric group and $T \coloneqq \{(ab) \mid a < b\}$ the set of *reflections*. The *length* of a permutation w is the number of inversions and is denoted $\ell(w)$. We write $u \xrightarrow{t} v$ if t is a reflection, $tu = v$, and $\ell(u) < \ell(v)$.

Bruhat graphs

- The **Bruhat graph** Ω_W is the edge-labeled, directed graph with vertex set W and edges $u \xrightarrow{t} v$ as above.
- We say $u \leq v$ in **Bruhat order** if there is path from u to v in the Bruhat graph. The Bruhat graph $\Omega_{u,v}$ is the induced subgraph of Ω_W on the elements of $[u, v]$.

Reflection orders

- A **reflection order** is a total ordering $\prec$ on the reflections T such that whenever $a < b < c$ we have
$$(ab) \prec (ac) \prec (bc) \quad \text{or} \quad (bc) \prec (ac) \prec (ab).$$
- An **increasing path** of length k in the Bruhat graph $\Omega_{u,v}$ is a path $u \xrightarrow{t_1} u_1 \xrightarrow{t_2} \cdots \xrightarrow{t_k} u_k = v$ such that $t_1 \prec t_2 \prec \cdots \prec t_k$ in the reflection order.

The $\widetilde{R}$**-polynomial** for the order $\prec$ is defined by
$$\widetilde{R}_{u,v}(q) = \sum_{k=0}^{\ell(v)-\ell(u)} (\#\text{ of length-}k\text{ increasing paths in }\Omega_{u,v}) \cdot q^k.$$
A priori, the $\widetilde{R}$-polynomial depends on a choice of reflection order, but in fact all choices give the same polynomial. It also seems like we need the edge-labels of $\Omega_{u,v}$, but...

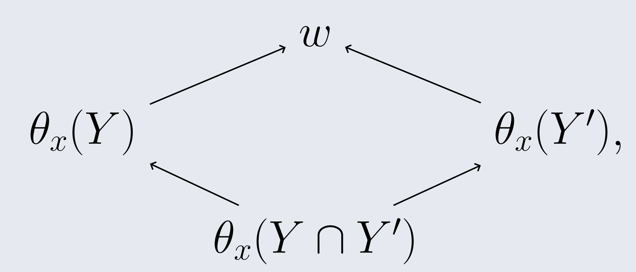
Combinatorial Invariance Conjecture (Lusztig 1980s; Dyer 1987)

If two Bruhat graphs $\Omega_{u,v}$ and $\Omega_{u',v'}$ are isomorphic as *unlabeled* directed graphs, then
$$\widetilde{R}_{u,v} = \widetilde{R}_{u',v'}.$$

The $\widetilde{H}$-polynomial

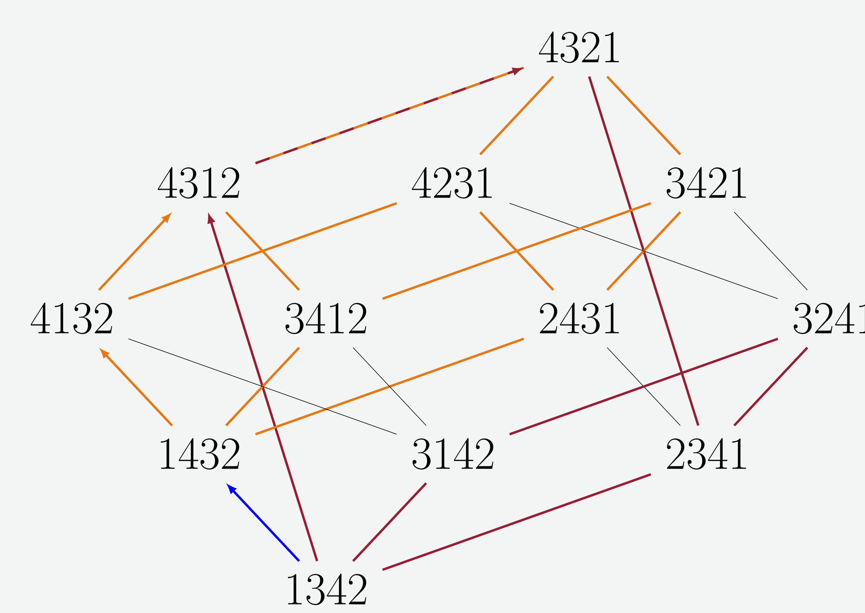
Hypercube clusters

- Let $I \subset [u, v]$ be an order ideal and let $x \in I$. Define
$$\mathcal{Y}_x = \{y \in [u, v] \mid x \to y,\ y \notin I\}$$
and
$$\mathcal{A}_x = \{Y \subset \mathcal{Y}_x \mid Y \text{ is an antichain in Bruhat order}\}.$$
We view $\mathcal{A}_x$ as a directed graph where $Y_1 \to Y_2$ if $Y_1 \subset Y_2$ and $|Y_2 \setminus Y_1| = 1$.
- A **strong hypercube cluster** at x (relative to I) consists of a function $\theta_x : \mathcal{A}_x \to [u, v]$ satisfying the following four properties:
  - $\theta_x(\varnothing) = x$.
  - If $y \in \mathcal{Y}_x$, then $\theta_x(\{y\}) = y$.
  - If $Y_1 \to Y_2$, then $\theta_x(Y_1) \to \theta_x(Y_2)$.
  - If $|Y| = |Y'| = |Y \cap Y'| + 1$ and we have a subgraph of $\Omega_{u,v}$ of the form



then $Y \cup Y'$ is an antichain and $\theta_x(Y \cup Y') = w$.

An example



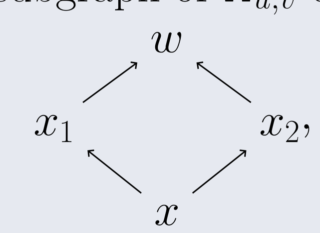
The Bruhat graph $\Omega_{u,v}$ with $u = 1342$ and $v = 4321$. The interval below $z = 1432$ (shown in blue) is a strong hypercube decomposition. The images of θ_x for $x = 1342$ and $x = 1432$ are shown in red and orange, respectively.

The pairs (x, Y) with $Y \in \mathcal{A}_x$ such that $\theta_x(Y) = v$ are
$$(1342, \{4312, 2341\}), \quad (1432, \{4132, 3412, 2431\}).$$
We find that $\widetilde{H}_{u,z,v} = q^2 \cdot 1 + q^3 \cdot q$.

Using (for instance) the reflection order
$$(34) \prec (24) \prec (23) \prec (14) \prec (13) \prec (12),$$
we find that $\widetilde{R}_{u,v} = q^4 + q^2 = \widetilde{H}_{u,z,v}$. The edges along the two increasing paths are shown with arrows.

Hypercube decompositions

- A **strong hypercube decomposition** of $[u, v]$ consists of an order ideal I satisfying the following properties:
  - $I = [u, z]$ for some $z \in [u, v]$.
  - For each x in I, there is a strong hypercube cluster $\theta_x : \mathcal{A}_x \to [u, v]$, relative to I.
  - If we have a subgraph of $\Omega_{u,v}$ of the form



with $x, x_1, x_2 \in I$, then $w \in I$ as well.

If $I = [u, z]$ is a strong hypercube decomposition of $[u, v]$, then the $\widetilde{H}$**-polynomial** is defined by
$$\widetilde{H}_{u,z,v} \coloneqq \sum_{x \in I} \sum_{\substack{Y \in \mathcal{A}_x \\ \theta_x(Y) = v}} q^{|Y|} \widetilde{R}_{u,x}.$$

Elementary intervals

- A Bruhat interval $[u, v]$ is **simple** if its atoms, viewed as points in $\mathbb{R}^n$, are affinely independent.
- A Bruhat interval $[u, v]$ is **elementary** if it is isomorphic as a poset to a simple interval.

An important example of elementary intervals are *lower intervals* $[e, v]$. These are the main intervals for which the CIC was heretofore known:

Theorem (Brenti 2004; du Cloux 2003)

The Combinatorial Invariance Conjecture holds for lower intervals.

Our main theorem is the first extension of this result to a broader class of intervals.

Main Theorem

If $[u, z]$ is a strong hypercube decomposition of an elementary interval $[u, v]$ in the symmetric group, then
$$\widetilde{R}_{u,v} = \widetilde{H}_{u,z,v}.$$
As a result, $\widetilde{R}_{u,v}$ can be computed using $\Omega_{u,v}$ as an unlabeled digraph. In other words, the Combinatorial Invariance Conjecture is true for elementary intervals of S_n.

Proof sketch

- If I is a strong hypercube decomposition of a simple interval, then there is a reflection order placing edge labels in I as early as possible.
- When we compute $\widetilde{R}_{u,v}$ using this order, the increasing paths will stay in I for as long as possible, eventually exiting I at a vertex $x \in I$.
- After exiting I, an increasing path will stay in the image of θ_x. The path it traces builds a unique element of $\mathcal{A}_x$.
- So increasing paths are counted by a choice of exit point x, an increasing path from u to x, and a choice of antichain $Y \in \mathcal{A}_x$ with $\theta_x(Y) = v$. These are exactly the objects counted by $\widetilde{H}_{u,z,v}$.

Future directions

Conjecture (Blundell, Buesing, Davies, Veličković, and Williamson)

There is a recurrence for Kazhdan–Lusztig polynomials for any interval of the symmetric group, which uses hypercube decompositions, and which would imply the CIC.

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