Weak Bruhat interval modules of the 0-Hecke algebras for genomic Schur functions

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0. Abstract

The genomic Schur function U_λ was introduced by Pechenik–Yong in the context of the K-theory of Grassmannians. Recently, Pechenik provided a positive combinatorial formula for the fundamental quasisymmetric expansion of U_λ in terms of increasing gapless tableaux. We construct an $H_m(0)$-module $\mathbf{G}_{\lambda;m}$ whose image under the quasisymmetric characteristic is the mth degree homogeneous component of U_λ by defining an $H_m(0)$-action on increasing gapless tableaux. We then assign a permutation to each increasing gapless tableau, and decompose $\mathbf{G}_{\lambda;m}$ into a direct sum of weak Bruhat interval modules by using this assignment. Furthermore, we determine the projective cover of each summand of the direct sum decomposition.

1. Preliminaries

Throughout, we fix a positive integer n.

1.1. The 0-Hecke algebra and the quasisymmetric characteristic

The 0-*Hecke algebra* $H_n(0)$ is the $\mathbb{C}$-algebra generated by $\pi_1, \pi_2, \dots, \pi_{n-1}$ subject to the following relations:

$$\begin{aligned} \pi_i^2 &= \pi_i \quad \text{for } 1 \le i \le n-1,\\ \pi_i\pi_{i+1}\pi_i &= \pi_{i+1}\pi_i\pi_{i+1} \quad \text{for } 1 \le i \le n-2,\\ \pi_i\pi_j &= \pi_j\pi_i \quad \text{if } |i-j| \ge 2. \end{aligned}$$

In 1979, Norton [4] classified all irreducible $H_n(0)$-modules $\mathbf{F}_\alpha$ ($\alpha \models n$).

We denote by

- $\mathcal{G}_0(H_n(0))$ the *Grothendieck group* of $H_n(0)$-**mod**,
- $\mathcal{G}(H_\bullet(0))$ the ring $(\bigoplus_{n\ge 0} \mathcal{G}_0(H_n(0))$, the induction product $\boxtimes)$,
- QSym the ring of quasisymmetric functions,
- F_α the fundamental quasisymmetric function attached to α.

In 1996, Duchamp et al.[1] introduced the ring isomorphism

$$\mathrm{ch} : \mathcal{G}(H_\bullet(0)) \to \mathrm{QSym}, \quad [\mathbf{F}_\alpha] \mapsto F_\alpha \quad (\alpha\text{: composition}),$$

called the *quasisymmetric characteristic*.

1.2. Weak Bruhat interval modules (abbr. WBIMs)

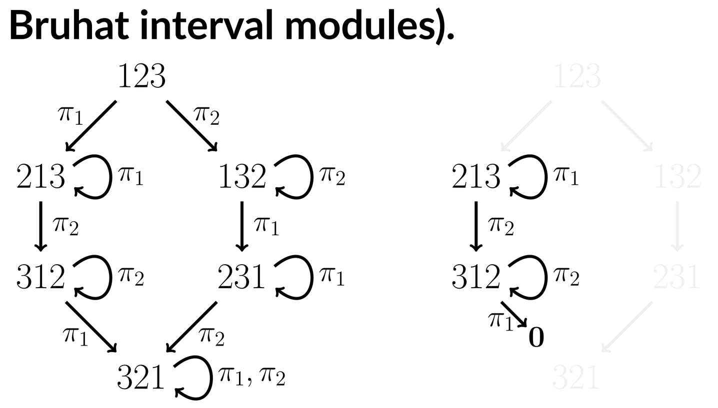
For $1 \le i \le n-1$, let $s_i := (i\ \ i+1) \in \mathfrak{S}_n$. For $\sigma, \rho \in \mathfrak{S}_n$,

- $\mathrm{Des}_L(\sigma) := \{i \in [n-1] \mid i \text{ is right of } i+1 \text{ in } \sigma(1)\,\sigma(2)\cdots\sigma(n)\}$,
- the *left weak Bruhat order* ($\preceq_L$): $\sigma \preceq_L s_i\sigma \iff i \notin \mathrm{Des}_L(\sigma)$,
- the *left weak Bruhat interval* $[\sigma,\rho]_L := \{\gamma \in \mathfrak{S}_n \mid \sigma \preceq_L \gamma \preceq_L \rho\}$.

Definition. ([2])
The *weak Bruhat interval module* $\mathsf{B}(\sigma,\rho)$ is the $H_n(0)$-module with
- underlying space: $\mathbb{C}[\sigma,\rho]_L$
- $H_n(0)$-action: for $\gamma \in [\sigma,\rho]_L$ and $1 \le i \le n-1$,

$$\pi_i \cdot \gamma := \begin{cases} \gamma & \text{if } i \in \mathrm{Des}_L(\gamma),\\ 0 & \text{if } i \notin \mathrm{Des}_L(\gamma) \text{ and } s_i\gamma \notin [\sigma,\rho]_L,\\ s_i\gamma & \text{if } i \notin \mathrm{Des}_L(\gamma) \text{ and } s_i\gamma \in [\sigma,\rho]_L. \end{cases}$$

Example (Weak Bruhat interval modules).



$\mathsf{B}(123,321)$ $\qquad$ $\mathsf{B}(213,312)$

Remark: Why are WBIMs interesting?
There are several quasisymmetric analogues of the Schur functions: For examples,
- Quasisymmetric Schur functions,
- Dual immaculate quasisymmetric functions,
- Extended Schur functions, and
- their automorphism images under ψ, ρ, ω.

In the last decade, several $H_n(0)$-modules have been constructed in association with these quasisymmetric functions. **Each indecomposable direct summand of the constructed $H_n(0)$-modules have a weak Bruhat interval module structure.**

1.3. Genomic Schur functions

The genomic Schur functions were introduced by Pechenik and Yong [6] in the context of the K-theory of Grassmannians.

- Increasing gapless tableaux:

$$\begin{smallmatrix}1&3&4\\2&5&7\\5&6\end{smallmatrix} \qquad \begin{smallmatrix}1&2&2\\3&4&5\\6&7\end{smallmatrix} \quad \begin{smallmatrix}1&2&4\\3&5&6\\3&7\end{smallmatrix} \qquad \begin{smallmatrix}1&2&3\\5&6&7\\7&8\end{smallmatrix}$$

(O) $\qquad$ non-increasing $\qquad$ non-gapless ($\nexists 4$)

- $\mathrm{IGLT}(\lambda)$: the set of all increasing gapless tableaux of shape λ
- $\mathrm{Des}(T) := \{i \in [n-1] \mid \exists\, i \text{ appears weakly above an } i+1 \text{ in } T\}$
- $\mathrm{comp}(T)$: the composition of n corresponding to $\mathrm{Des}(T)$

$$T = \begin{smallmatrix}1&3&4\\2&5&7\\5&6\end{smallmatrix} \Rightarrow \mathrm{Des}(T) = \{1,3,4,5\} \Rightarrow \mathrm{comp}(T) = (1,2,1,1,2)$$

Definition. ([6]) (for the F-expansion, see [5])
For any $\lambda \vdash n$, *the genomic Schur function attached to* λ is

$$U_\lambda := \sum_{T \in \mathrm{IGLT}(\lambda)} F_{\mathrm{comp}(T)}.$$

For $1 \le m \le n$, let

$$\left[\begin{aligned} \mathrm{IGLT}(\lambda)_m &:= \{T \in \mathrm{IGLT}(\lambda) \mid \max(T) = m\},\\ U_{\lambda;m} &:= \textstyle\sum_{T \in \mathrm{IGLT}(\lambda)_m} F_{\mathrm{comp}(T)}. \end{aligned}\right.$$

Example (Genomic Schur functions).

$$\mathrm{IGLT}((2,1,1)) = \left\{ \begin{smallmatrix}1&2\\3\\4\end{smallmatrix}, \begin{smallmatrix}1&3\\2\\4\end{smallmatrix}, \begin{smallmatrix}1&4\\2\\3\end{smallmatrix}, \begin{smallmatrix}1&2\\2\\3\end{smallmatrix}, \begin{smallmatrix}1&3\\2\\3\end{smallmatrix} \right\}.$$

Thus, $U_{(2,1,1)} = U_{(2,1,1);4} + U_{(2,1,1);3} = (F_{(2,1,1)} + F_{(1,2,1)} + F_{(1,1,2)}) + 2F_{(1,1,1)}$.

Remark: It is impossible to construct an indecomposable $H_m(0)$-module M satisfying $\mathrm{ch}([M]) = U_{\lambda;m}$ for some $\lambda \vdash n$ and $1 \le m \le n$.
In 2002, Duchamp–Hivert–Thibon showed that there is no indecomposable $H_m(0)$-module M such that $\mathrm{ch}([M]) = 2F_\alpha$. Since $U_{(2,1,1);3} = 2F_{(1,1,1)}$, there does not exist $H_3(0)$-module M satisfying $\mathrm{ch}([M]) = U_{(2,1,1);3}$.

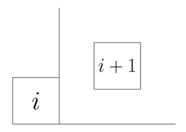
Goal and Strategy

Goal.
Construct an $H_m(0)$-module $\mathbf{G}_{\lambda;m}$ such that
(i) $\mathrm{ch}(\mathbf{G}_{\lambda;m}) = U_{\lambda;m}$,
(ii) $\mathbf{G}_{\lambda;m}$ can be decomposed into weak Bruhat interval submodules.

Strategy.
Step 1. Define $\mathbf{G}_{\lambda;m}$ by defining an $H_m(0)$-action on $\mathbb{C}\mathrm{IGLT}(\lambda)_m$.
Step 2. Define an equivalence relation $\sim_{\lambda;m}$ on $\mathrm{IGLT}(\lambda)_m$ and decompose $\mathbf{G}_{\lambda;m}$ into submodules considering $\sim_{\lambda;m}$ (see (†)).
Step 3. For each equivalence class E, prove that there exist unique *source tableau* T_E and unique *sink tableau* T'_E.
Step 4. For each equivalence class E, define a map $\mathsf{read} : E \to \mathfrak{S}_m$ and show that

$$\mathbf{G}_E \cong \mathsf{B}(\mathsf{read}(T_E), \mathsf{read}(T'_E)).$$

2. The $H_m(0)$-module $\mathbf{G}_{\lambda;m}$

- *Attacking descents*: $i \in \mathrm{Des}(T)$ such that $\begin{smallmatrix} i \\ i+1 \end{smallmatrix}$ or  ($i+1$ is placed weakly above i)

For instance,

$$\text{if} \quad T = \begin{smallmatrix}1&3&4\\2&5&7\\5&6\end{smallmatrix}, \quad \text{then} \quad \begin{cases} 2, 6 \text{ are not descents of } T,\\ 1, 3, 5 \text{ are attacking descents of } T,\\ 4 \text{ is a non attacking descent of } T. \end{cases}$$

Definition. ([3])
For $1 \le m \le n$, $\mathbf{G}_{\lambda;m}$ is the $H_m(0)$-module with
- underlying space: $\mathbb{C}\,\mathrm{IGLT}(\lambda)_m$
- $H_m(0)$-action: for $T \in \mathrm{IGLT}(\lambda)_m$ and $1 \le i \le m-1$,

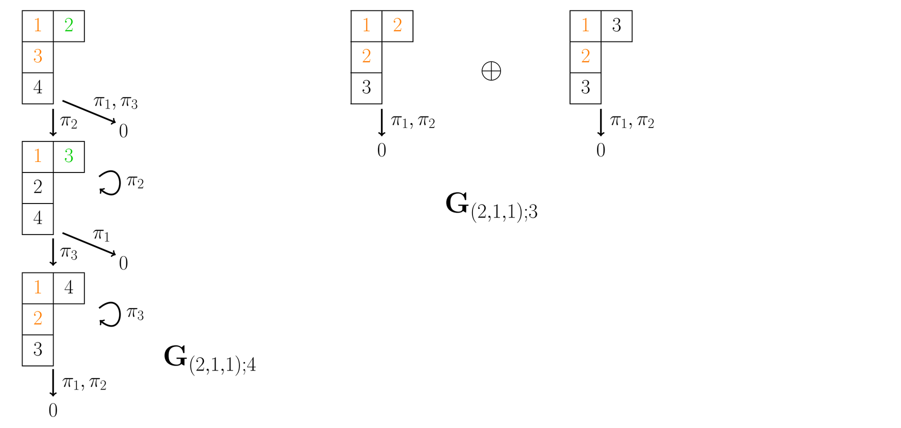
$$\pi_i \cdot T := \begin{cases} T & \text{if } i \text{ is not a descent of } T,\\ 0 & \text{if } i \text{ is an attacking descent of } T,\\ s_i \cdot T & \text{if } i \text{ is a non-attacking descent of } T. \end{cases}$$

Theorem. ([3])
(1) $\mathbf{G}_{\lambda;m}$ is a well-defined $H_m(0)$-module.
(2) $\mathrm{ch}([\mathbf{G}_{\lambda;m}]) = U_{\lambda;m}$. Consequently, $\sum_{1 \le m \le n} \mathrm{ch}([\mathbf{G}_{\lambda;m}]) = U_\lambda$.

Example ($H_m(0)$-action on $\mathbf{G}_{\lambda;m}$).
(1) For the above T, we have

$$\pi_i \cdot T = T \ \text{ for } i = 2, 6, \quad \pi_i \cdot T = 0 \ \text{ for } i = 1, 3, 5, \quad \pi_4 \cdot T = \begin{smallmatrix}1&3&5\\2&4&7\\4&6\end{smallmatrix}.$$

(2)



3. A direct sum decomposition of $\mathbf{G}_{\lambda;m}$

Hereafter, we fix a $T \in \mathrm{IGLT}(\lambda)_m$ unless otherwise stated.

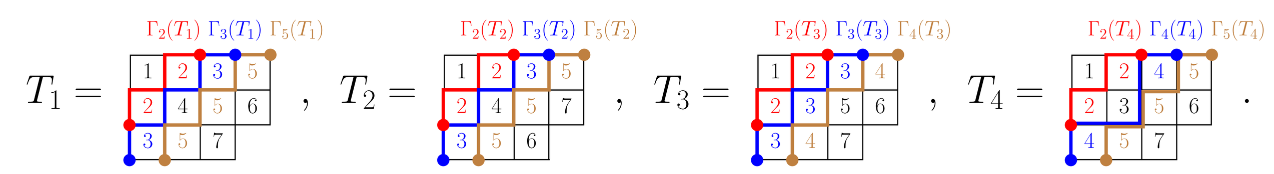
- $\mathcal{I}(T) := \{i \in [1,m] \mid |T^{-1}(i)| > 1\}$.
- For $i \in \mathcal{I}(T)$, define the lattice path $\Gamma_i(T)$ as the following example:

Example ($\mathcal{I}(T), \Gamma_i(T), \sim_{\lambda;m}$).

$$\text{For } T_1 = \begin{smallmatrix}1&2&3&5\\2&4&5&6\\3&5&7\end{smallmatrix}, \quad T_2 = \begin{smallmatrix}1&2&3&5\\2&4&5&7\\3&5&6\end{smallmatrix}, \quad T_3 = \begin{smallmatrix}1&2&3&4\\2&3&5&6\\3&4&7\end{smallmatrix}, \quad T_4 = \begin{smallmatrix}1&2&4&5\\2&3&5&6\\4&5&7\end{smallmatrix},$$

$\mathcal{I}(T_1) = \{2,3,5\}, \quad \mathcal{I}(T_2) = \{2,3,5\}, \quad \mathcal{I}(T_3) = \{2,3,4\}, \quad \mathcal{I}(T_4) = \{2,4,5\}.$

We define $\Gamma_i(T_k)$'s as follows



- Define an equivalence relation $\sim_{\lambda;m}$ on $\mathrm{IGLT}(\lambda)_m$ by

$$T_1 \sim_{\lambda;m} T_2 \Leftrightarrow \left\{ \left(\Gamma_i(T_1), T_1^{-1}(i)\right) \middle| i \in \mathcal{I}(T_1) \right\} = \left\{ \left(\Gamma_i(T_2), T_2^{-1}(i)\right) \middle| i \in \mathcal{I}(T_2) \right\}.$$

In the previous example, $T_1 \sim_{(4,3,2);5} T_2$ and $T_1 \not\sim_{(4,3,2);5} T_k$ for $k = 3, 4$.

- Let $\mathcal{E}_{\lambda;m}$ be the set of equivalence classes of $\mathrm{IGLT}(\lambda)_m$ w.r.t. $\sim_{\lambda;m}$.

Theorem. ([3])
For $E \in \mathcal{E}_{\lambda;m}$, $\mathbb{C}E$ is closed under the $H_m(0)$-action.

- Let $\mathbf{G}_E$ be the $H_m(0)$-submodule of $\mathbf{G}_{\lambda;m}$ whose underlying space is $\mathbb{C}E$.

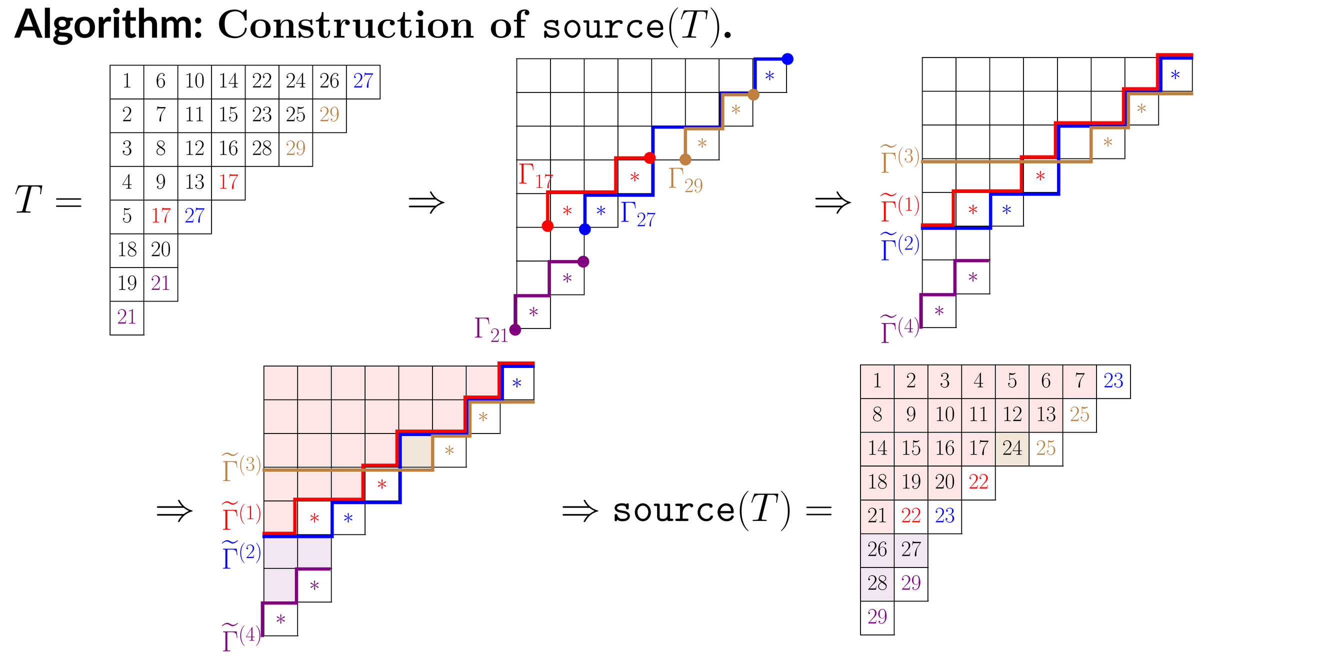
Then, we have

$$\mathbf{G}_{\lambda;m} = \bigoplus_{E \in \mathcal{E}_{\lambda;m}} \mathbf{G}_E. \qquad (\dagger)$$

4. Source and sink tableaux

Definition.
- T is a *source tableau* if $\nexists\, (T', i) \in (\mathrm{IGLT}(\lambda)_m \setminus \{T\}) \times [m-1]$ s.t. $\pi_i \cdot T' = T$.
- T is a *sink tableau* if $\nexists\, (T', i) \in (\mathrm{IGLT}(\lambda)_m \setminus \{T\}) \times [m-1]$ s.t. $\pi_i \cdot T = T'$.

Algorithm: Construction of $\mathtt{source}(T)$.



In a similar way, we also construct $\mathtt{sink}(T)$.

Theorem. ([3])
Given $T \in \mathrm{IGLT}(\lambda)_m$, let $E \in \mathcal{E}_{\lambda;m}$ be the equivalence class containing T.
(i) $\mathtt{source}(T)$ is the unique source tableau in E, and $\qquad$ (denote by T_E)
(ii) $\mathtt{sink}(T)$ is the unique sink tableau in E. $\qquad$ (denote by T'_E)

5. A WBIM description of $\mathbf{G}_E$

For $T \in E$, we define *the standardized reading word* $\mathsf{read}(T)$ *of* T as follows:

$$T_E = \begin{smallmatrix}1&2&3&4&5&11&12&14&15\\6&7&8&10&11&14\\9&10\\13&14\end{smallmatrix} \Rightarrow \begin{smallmatrix}\mathtt{H}_1&\mathtt{H}_1&\mathtt{H}_1&\mathtt{H}_1&\mathtt{H}_1&\mathtt{H}_3&\mathtt{H}_3&\mathtt{H}_4&\mathtt{H}_4\\\mathtt{H}_2&\mathtt{H}_2&\mathtt{H}_2&\mathtt{H}_3&\mathtt{H}_3&\mathtt{H}_4\\\mathtt{H}_3&\mathtt{H}_3\\\mathtt{H}_4&\mathtt{H}_4\end{smallmatrix}$$

(Decompose $\mathtt{yd}(\lambda)$ considering $\mathrm{Des}(T_E)$)

$$T_E = \begin{smallmatrix}1&2&3&4&5&11&12&14&15\\6&7&8&10&11&14\\9&10\\13&14\end{smallmatrix} \Rightarrow \mathsf{read}(T_E) = 5\,4\,3\,2\,1\,8\,7\,6\,12\,11\,10\,9\,15\,14\,13 \in \mathfrak{S}_{15}.$$

$$T = \begin{smallmatrix}1&2&4&5&10&12&13&14&15\\3&6&7&9&12&14\\8&9\\11&14\end{smallmatrix} \Rightarrow \mathsf{read}(T) = 10\,5\,4\,2\,1\,7\,6\,3\,13\,12\,9\,8\,15\,14\,11 \in \mathfrak{S}_{15}.$$

Theorem. ([3])
As $H_m(0)$-modules, $\mathbf{G}_E \cong \mathsf{B}(\mathsf{read}(T_E), \mathsf{read}(T'_E))$.

6. The projective cover of $\mathbf{G}_E$

Let $\boldsymbol{\alpha} = \alpha^{(1)} \oplus \cdots \oplus \alpha^{(k)}$ be a formal sum of compositions with $\sum_{i=1}^k |\alpha^{(i)}| = m$. In 2016, Huang defined a projective $H_m(0)$-module $\mathbf{P}_{\boldsymbol{\alpha}}$. We note that

$$\mathbf{P}_{\boldsymbol{\alpha}} \cong \mathsf{B}\big(w_0(\mathfrak{S}_{\mathrm{set}(\boldsymbol{\alpha}_\bullet)^c}), w_0(\mathfrak{S}_m)w_0(\mathfrak{S}_{\mathrm{set}(\boldsymbol{\alpha}_\odot)})\big),$$

where $\boldsymbol{\alpha}_\bullet$ (resp. $\boldsymbol{\alpha}_\odot$) is the concatenation (resp. near concatenation) of $\alpha^{(1)}, \dots, \alpha^{(k)}$ and for $\beta \models m$, $w_0(\mathfrak{S}_{\mathrm{set}(\beta)})$ is the longest element of the parabolic subgroup $\mathfrak{S}_{\mathrm{set}(\beta)}$ of $\mathfrak{S}_m$.

Lemma. ([3])
For $\boldsymbol{\alpha} = \alpha^{(1)} \oplus \cdots \oplus \alpha^{(k)}$, $[w_0(\boldsymbol{\alpha}^c_\odot), w_0 w_0(\boldsymbol{\alpha}_\odot)]_L \setminus [w_0(\boldsymbol{\alpha}^c_\bullet), w_0(\boldsymbol{\alpha}^c_\odot)]_L \subseteq \mathrm{rad}(\mathbf{P}_{\boldsymbol{\alpha}})$.

So, $\forall \gamma \in [w_0(\boldsymbol{\alpha}^c_\odot), w_0 w_0(\boldsymbol{\alpha}_\odot)]_L$, $\mathbf{P}_{\boldsymbol{\alpha}}$ is the projective cover of $\mathsf{B}(w_0(\boldsymbol{\alpha}^c_\bullet), \gamma)$. Using this result, we determine the projective cover of $\mathbf{G}_E$ for all $E \in \mathcal{E}_{\lambda;m}$.

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