Weak Bruhat interval modules of the 0-Hecke algebras for genomics Schur functions

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0. Abstract

The genomic Schur function $E_\lambda$ was introduced by Pechenik-Yang in the context of the K-theory of Grassmannians. Recently, Pechenik provided a positive combinatorial formula for the fundamental quasipolynomial expansion of $E_\lambda$ in terms of increasing gapless tableaux. We construct an $H_\nu(0)$-module $\mathcal{G}_\lambda\nu$ whose image under the quasisymmetric characteristic is the weak degeneracy homogenous component of $E_\lambda$, by defining an $H_\nu(0)$-action on increasing gapless tableaux. We then assign a permutation to each increasing gapless tableau, and decompose $\mathcal{G}_\lambda\nu$ into a direct sum of weak Bruhat interval modules by using this assignment. Furthermore, we determine the projective cover of each summand of the direct sum decomposition.

1. Preliminaries

1.1. The 0-Hecke algebra and the quasisymmetric characteristic

The 0-Hecke algebra $H_0$ is the C-algebra generated by $p_1, p_2, \ldots, p_n$ subject to the following relations:

- $p_i^2 = p_i$ for all $1 \leq i \leq n$.
- $p_i p_j = p_j p_i$ if $|j - i| > 1$.
- $p_i p_{i+1} p_i = p_{i+1} p_i p_{i+1}$.
- $p_i^2 = p_{i+1}$.

In $1979$, Norton [4] classified all irreducible $H_0$-modules $P_\alpha$ ($\alpha \in \mathfrak{S}_n$). We denote by

- $\mathbb{C}[(H_0, H_0, \nu)]$ the Grothendieck group of the $H_0$-mod.
- $\mathbb{C}[(H_0, H_0, \nu)]$ the ring $\bigoplus_\nu (H_0, H_0, \nu)$, the induction product $\oplus$.

2. The $H_\nu(0)$-module $\mathcal{G}_\lambda\nu$

2.1. Genomic Schur functions

The genomic Schur functions were introduced by Pechenik and Yang [6] in the context of the K-theory of Grassmannians.

- Increasing gapless tableaux:

$$\begin{array}{ll}
0 & \text{non-increasing} \\
1 & \text{non-gapless} \\
2 & \text{gapless}
\end{array}$$

- $\mathcal{G}(\lambda)$: the set of all increasing gapless tableaux of shape $\lambda$.

- $\nu(T) = (\nu = [n-1]| 3 \nu$ appears weakly above $1 + 1 + 1)$

- $\nu(T)$: the composition of $\nu$ corresponding to $\nu(T)$.

Definition. ([6]) for the $F$-expansion, see [5].

For any $\nu \in \mathfrak{S}_n$, the genomic Schur function attached to $\nu$ is

$$U_\nu = \sum_{T \in \mathcal{G}(\nu)} F_{\nu(T)}.$$ 

Example (Genomic Schur functions).

$$\begin{array}{c|c|c|c|c|c}
\nu & F_{\nu(T)} & \vdots & F_{\nu(T)} & \vdots & F_{\nu(T)} \\
\hline
2 & T_{1,2} & T_{2,1} & T_{1,2} & T_{2,1} & T_{1,2} \\
\end{array}$$

3. A direct sum decomposition of $G_{\lambda\nu}$

Hereafter, we fix a $T \in \mathcal{G}(\lambda)$ otherwise stated.

- $\mathcal{G}(\lambda)$: the unshuffle product $\mathcal{G}(\lambda)$ of $\mathcal{G}(\lambda)$.

- $\mathcal{G}(\lambda) \cdot \mathcal{G}(\nu)$: the weak Bruhat interval $[\lambda, \nu]$.

- The weak Bruhat interval module $B(\sigma, \rho)$ is the $H_\nu(0)$-module with underlying space $\mathcal{G}(\sigma)$.

- Definition. ([6]) The weak Bruhat interval module $B(\sigma, \rho)$ is the $H_\nu(0)$-module with underlying space $\mathcal{G}(\sigma)$.

- $\mathcal{G}(\lambda)$-action: for $\gamma \in \mathcal{G}(\lambda)$ subject to $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ with $\lambda_i \geq 0$.

- $\nu(\gamma)$ is defined as the weak Bruhat interval $[\lambda, \nu]$.

Goal and Strategy

Goal.

Construct an $H_\nu(0)$-module $\mathcal{G}_\lambda\nu$ such that

(i) $\mathcal{G}_\lambda\nu = \mathcal{G}(\lambda)$,

(ii) $\mathcal{G}_\lambda\nu$ can be decomposed into weak Bruhat interval submodules.

Strategy.

Step 1. Define $\mathcal{G}_{\lambda\nu}$ by defining an $H_\nu(0)$-action on $\mathcal{G}(\lambda)$.

Step 2. Define an equivalence relation $\sim_\nu$ on $\mathcal{G}(\lambda)$ and decompose $\mathcal{G}_\lambda\nu$ into submodules considering $\sim_\nu$ (see [3]).

Step 3. For each equivalence class $\mathcal{E}$, prove that there exist unique tableau $T_\mathcal{E}$ and unique sink tableau $T'_\mathcal{E}$ for each equivalence class $\mathcal{E}$.

Step 4. For each equivalence class $\mathcal{E}$, define a map read $\mathcal{G}(\lambda) \to \mathcal{G}(\lambda)$, and show that

$$\mathcal{G}_\lambda\nu = \bigoplus_{\mathcal{E}} \mathcal{G}(\lambda).$$

4. Source and sink tableaux

Definition.

- A source tableau $T'$ is $\nu(\lambda)$ such that $\nu(T') \in (\nu(T') \cup \nu(T')^+)$ and $T' = T$.

- A sink tableau $\nu(T)$ such that $\nu(T) \in (\nu(T) \cup \nu(T)^+)$ and $T = T$.

Algorithm: Construction of source($T_\mathcal{E}$).

- $\nu(T_\mathcal{E}) = \nu(T)$ if $\nu(T)$ is not a descent of $T_\mathcal{E}$.
- $\nu(T_\mathcal{E}) = \nu(T)$ if $\nu(T)$ is an attacking descent of $T_\mathcal{E}$.

5. A WBM description of $G_E$

For $E \in \mathcal{G}(\lambda)$, we define the standardival reading word $\Gamma(E)$ as follows:

- $\Gamma(E) = \{i \in [m] | \nu^{\text{rev}}(i) > 1\}$.

- For $E \in \mathcal{G}(\lambda)$, define the lattice path $\Gamma(E)$ as the following example.

Example: $\Gamma(E)$.

$\begin{array}{c}
\Gamma(E) = \{1, 2, 3, 4\} \\
\end{array}$

$\begin{array}{c}
\nu(T_\mathcal{E}) = \nu(T) \in (\nu(T) \cup \nu(T)^+) \\
\end{array}$

Theorem. ([3]) $G_E$ is a well-defined $H_\nu(0)$-module.

6. The projective cover of $G_E$

Let $\alpha = \alpha(0, \alpha_0, \alpha_1) \in \nu(\lambda)$ be a formal sum of compositions with $\sum \alpha_i(0, \alpha_0, \alpha_1) = m$. In 2016, Huang defined a projective $H_\nu(0)$-module $\mathcal{P}_\alpha$. We note that

$$\mathcal{P}_\alpha \cong \bigoplus_{w \in \mathcal{S}_\alpha} \nu(w) \mathcal{G}(\lambda).$$

where $\mathcal{S}_\alpha$ (resp. $\mathcal{S}_\alpha^+$) is the concatenation (resp. near concatenation) of $\alpha^{(0)}$, $\alpha_0$ and for $\beta \in \mathcal{S}_\alpha$, $w(\mathcal{S}_\alpha)$ is the longest element of the parabolic subgroup $\mathcal{S}(\alpha)$ of $\mathcal{S}_\alpha$. In this example, $\mathcal{S}_\alpha = \{1, 2, 3, 4\}$.

Let $\mathcal{E}_\mathcal{S}(\alpha)$ be the set of equivalence classes of $\lambda$-cells w.r.t. $\alpha$.

Lemma. ([3])

For $\alpha = \alpha(0, \alpha_0, \alpha_1), \mathcal{P}_\alpha$ is the projective cover of $G_E$ for $E \in \mathcal{E}_\mathcal{S}(\alpha)$.

Using this result, we determine the projective cover of $G_E$ for $E \in \mathcal{E}_\mathcal{S}(\alpha)$.

References


