Stable-Limit Non-symmetric Macdonald Functions in Type A

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Introduction

The ring of symmetric functions has a central place in algebraic combinatorics and representation theory. A generalization of the ring of symmetric functions, called the ring of **almost-symmetric functions**, has appeared in recent years implicitly in the work of Carlsson-Mellit on the Shuffle Theorem [1] as well as in the work of Ion-Wu with their +stable-limit DAHA [4]. In this work we construct a weight basis of almost-symmetric functions $\mathcal{P}_{as}^+$ for the limit Cherednik operators of Ion and Wu. This basis, consisting of the **stable-limit non-symmetric Macdonald functions**, is built from classical non-symmetric Macdonald polynomials in type GL along with Hall-Littlewood symmetric function creation operators. We investigate some of the combinatorial and algebraic properties that this basis possesses. A full version of the paper corresponding to this poster is on the arXiv [5].

Background

Symmetric Functions

The ring of **symmetric functions**, Λ, is the subalgebra of $\varprojlim \mathbb{Q}(q,t)[x_1,\ldots,x_n]^{\mathfrak{S}_n}$ consisting of elements with bounded degree. Here $\mathfrak{S}_n$ is the symmetric group on $\{1,\ldots,n\}$. Distinguished bases for Λ include the Schur functions $s_\lambda[X]$, the dual Hall-Littlewood symmetric functions $\mathcal{P}_\lambda[X;t]$, and the symmetric Macdonald functions $P_\lambda[X;q,t]$.

Cherednik Operators and Non-symmetric Macdonald Polynomials

Consider the following operators on the space of Laurent polynomials $\mathbb{Q}(q,t)[x_1^{\pm 1},\ldots,x_n^{\pm 1}]$:

- $\omega_n f(x_1,\ldots,x_n) := f(q^{-1}x_n, x_1,\ldots,x_{n-1})$
- $T_i f := s_i f + (1-t)x_i \frac{f - s_i f}{x_i - x_{i+1}}$
- $Y_i^{(n)} := t^{-(i-1)} T_{i-1}\cdots T_1 \omega_n^{-1} T_{n-1}^{-1}\cdots T_i^{-1}$.

The $Y_1^{(n)},\ldots,Y_n^{(n)}$ are the **Cherednik operators**. The non-symmetric Macdonald polynomials E_μ introduced by Cherednik [2] for $\mu \in \mathbb{Z}^n$ are the unique basis of $\mathbb{Q}(q,t)[x_1^{\pm 1},\ldots,x_n^{\pm 1}]$ consisting of simultaneous eigenvectors for the Cherednik operators with a natural normalization condition.

Almost-symmetric Functions and Limit Cherednik Operators

The ring of **almost-symmetric functions**, $\mathcal{P}_{as}^+$, is given by $\mathbb{Q}(q,t)[x_1,x_2,\ldots] \otimes \Lambda$. These are the bounded degree functions $f = f(x_1,x_2,\ldots)$ which are *eventually* symmetric i.e. there exists some $k \geq 0$ so that $s_i(f) = f$ for all $i > k$.

Ion and Wu [4] defined the **limit Cherednik operators** for all $i \geq 1$ using a new notion of convergence as

$$\mathcal{Y}_i := \lim_m t^{m-i+1} T_{i-1}\cdots T_1 \rho \omega_m^{-1} T_{m-1}^{-1}\cdots T_i^{-1}\pi_m.$$

Here $\pi_m : \mathcal{P}_{as}^+ \to \mathbb{Q}(q,t)[x_1,\ldots,x_m]$ is the natural projection and $\rho : \mathcal{P}_{as}^+ \to x_1\mathcal{P}_{as}^+$ is the projection which annihilates monomials not divisible by x_1.

Example Calculation

$$\begin{aligned}\mathcal{Y}_1(x_1) &= \lim_m t^m \rho\omega_m^{-1} T_{m-1}^{-1}\cdots T_1^{-1} x_1 \\ &= \lim_m t^m \rho \omega_m^{-1} t^{-(m-1)} x_m \\ &= \lim_m t\rho(qx_1) \\ &= qtx_1.\end{aligned}$$

Problem 1

Is there a $\mathcal{Y}$-weight basis of $\mathcal{P}_{as}^+$ and if so how do the weight vectors relate to finite variable non-symmetric Macdonald polynomials?

Theorem 1 [BW 23]

There exists a homogeneous $\mathcal{Y}$-weight basis $\widetilde{E}_{(\mu|\lambda)}$ of $\mathcal{P}_{as}^+$ indexed by pairs of reduced compositions μ and partitions λ, with the following properties:

- $\widetilde{E}_{(\mu|\emptyset)} = \lim_m E_{\mu * 0^m}$
- $\widetilde{E}_{(\emptyset|\lambda)} = (1-t)^{\ell(\lambda)} P_\lambda[X;q^{-1},t]$
- $\partial_-^{(r)}\left(\widetilde{E}_{(\mu_1,\ldots,\mu_r|\lambda_1,\ldots,\lambda_k)}\right) = \widetilde{E}_{(\mu_1,\ldots,\mu_{r-1}|\mu_r,\lambda_1,\ldots,\lambda_k)}$ whenever $\mu_r \geq \lambda_1$ and $\mu_{r-1} \neq 0$.

Examples

Here we give a few basic examples of stable-limit non-symmetric Macdonald functions expanded in the dual Hall-Littlewood basis $\mathcal{P}_\lambda$ and their corresponding weights.

- $\widetilde{E}_{(\emptyset|\emptyset)} = 1;$ weight $\widetilde{\alpha}_{(\emptyset|\emptyset)} = (0,0,\ldots)$
- $\widetilde{E}_{(1|\emptyset)} = x_1;$ weight $\widetilde{\alpha}_{(1|\emptyset)} = (qt,0,\ldots)$
- $\widetilde{E}_{(\emptyset|1)} = \mathcal{P}_1[X];$ weight $\widetilde{\alpha}_{(\emptyset|1)} = (0,0,\ldots)$
- $\widetilde{E}_{(2|\emptyset)} = x_1^2 + \frac{q^{-1}}{1-q^{-1}t} x_1\mathcal{P}_1[x_2+\ldots];$ weight $\widetilde{\alpha}_{(2|\emptyset)} = (q^2t,0,\ldots)$
- $\widetilde{E}_{(\emptyset|2)} = \mathcal{P}_2[X] + \frac{q^{-1}}{1-q^{-1}t}\mathcal{P}_{1,1}[X];$ weight $\widetilde{\alpha}_{(\emptyset|2)} = (0,0,\ldots)$
- $\widetilde{E}_{(1,1|1)} = x_1x_2\mathcal{P}_1[x_3+\ldots];$ weight $\widetilde{\alpha}_{(1,1|1)} = (qt^3,qt^2,0,\ldots)$
- $\widetilde{E}_{(1|1,1)} = x_1\mathcal{P}_{1,1}[x_2+\cdots];$ weight $\widetilde{\alpha}_{(1|1,1)} = (qt^3,0,\ldots)$

Main Properties

- $\widetilde{E}_{(\mu|\lambda)}$ is symmetric in the variables $\{x_{\ell(\mu)+1}, x_{\ell(\mu)+2},\ldots\}$.
- $\widetilde{E}_{(\mu|\lambda)}$ is homogeneous of degree $|\mu|+|\lambda|$.
- $\mathcal{Y}_i(\widetilde{E}_{(\mu|\lambda)}) = \widetilde{\alpha}_{(\mu|\lambda)}(i)\widetilde{E}_{(\mu|\lambda)}$ where
$$\widetilde{\alpha}_{(\mu|\lambda)}(i) = \begin{cases}\widetilde{\alpha}_{\mu*\lambda}(i) = q^{\mu_i}t^{\ell(\mu)+\ell(\lambda)+1-\beta_{\mu*\lambda}(i)} & i \leq \ell(\mu), \mu_i \neq 0 \\ 0 & \text{otherwise}\end{cases}$$
where
$$\beta_\nu(i) := \#\{j : 1 \leq j \leq i\ , \nu_j \leq \nu_i\} + \#\{j : i < j \leq n\ , \nu_i > \nu_j\}.$$
- For a reduced composition μ there is an HHL-like combinatorial formula given by
$$\widetilde{E}_{(\mu|\emptyset)} = \sum_{\substack{\lambda \text{ partition} \\ |\lambda| \leq |\mu|}} m_\lambda[x_{n+1}+\cdots] \sum_{\substack{\sigma:\mu*0^{\ell(\lambda)} \to [n+\ell(\lambda)] \\ \text{non-attacking} \\ \forall i=1,\ldots,\ell(\lambda) \\ \lambda_i = |\sigma^{-1}(n+i)|}} x_1^{|\sigma^{-1}(1)|}\cdots x_n^{|\sigma^{-1}(n)|}\widetilde{\Gamma}(\widehat{\sigma})$$
where
$$\widetilde{\Gamma}(\widehat{\sigma}) := q^{-\operatorname{maj}(\widehat{\sigma})} t^{\operatorname{coinv}(\widehat{\sigma})} \prod_{\substack{u \in dg'(\mu*0^{\ell(\lambda)}) \\ \widehat{\sigma}(u) \neq \widehat{\sigma}(d(u)) \\ u \text{ not in row } 1}} \left(\frac{1-t}{1-q^{-(\operatorname{lg}(u)+1)}t^{(a(u)+1)}}\right) \prod_{\substack{u \in dg'(\mu*0^{\ell(\lambda)}) \\ \widehat{\sigma}(u) \neq \widehat{\sigma}(d(u)) \\ u \text{ in row } 1}} (1-t)\,.$$

Problem 2

As can be seen from the previous examples, the $\mathcal{Y}$-weight spaces of $\mathcal{P}_{as}^+$ are not 1-dimensional. In particular, each $\mathcal{Y}_i$ annihilates Λ. Can we find another operator which commutes with the $\mathcal{Y}$-action which distinguishes between those $\mathcal{Y}$-weight vectors with identical weight?

Theorem 2 [BW 23]

There exists an operator $\Psi_{p_1} \in End_{\mathbb{Q}(q,t)}(\mathcal{P}_{as}^+)$ constructed as a limit of operators from finite variable DAHAs which is diagonal in the $\widetilde{E}_{(\mu|\lambda)}$ basis and distinguishes between those $\mathcal{Y}$-weight vectors with identical $\mathcal{Y}$-weight.

Ψ_{p_1} and 1-Dimensional Weight Spaces in $\mathcal{P}_{as}^+$

Explicitly, the operator Ψ_{p_1} is given by

$$\Psi_{p_1}(f) = \lim_m t^m (Y_1^{(m)} + \ldots + Y_m^{(m)})\pi_m(f).$$

On the stable-limit non-symmetric Macdonald function basis Ψ_{p_1} acts by

$$\Psi_{p_1}(\widetilde{E}_{(\mu|\lambda)}) = \kappa_{\operatorname{sort}(\mu*\lambda)}(q,t)\widetilde{E}_{(\mu|\lambda)}$$

where for a partition $\nu = (\nu_1,\ldots,\nu_r)$

$$\kappa_\nu(q,t) = \sum_{i=1}^r q^{\nu_i}t^i + \frac{t^{r+1}}{1-t}.$$

Let $\widetilde{Y}$ denote the subalgebra of $End_{\mathbb{Q}(q,t)}(\mathcal{P}_{as}^+)$ generated by the operator Ψ_{p_1} and the action of the operators $\mathcal{Y}_i$. Then $\mathcal{P}_{as}^+$ has a basis of $\widetilde{Y}$-weight vectors all with distinct weights.

Examples

The following stable-limit non-symmetric Macdonald functions all have degree 10, are symmetric in the variables $\{x_2,x_3,\ldots\}$, and have the same $\mathcal{Y}$-weight $(q^2t^3,0,\ldots)$. However, they have different Ψ_{p_1} eigenvalues.

- $\Psi_{p_1}(\widetilde{E}_{(2|6,2)}) = (q^6t + q^2t^2 + q^2t^3 + \frac{t^4}{1-t})\widetilde{E}_{(2|6,2)}$
- $\Psi_{p_1}(\widetilde{E}_{(2|5,3)}) = (q^5t + q^3t^2 + q^2t^3 + \frac{t^4}{1-t})\widetilde{E}_{(2|5,3)}$
- $\Psi_{p_1}(\widetilde{E}_{(2|4,4)}) = (q^4t + q^4t^2 + q^2t^3 + \frac{t^4}{1-t})\widetilde{E}_{(2|4,4)}$

Future Questions

- Are there analogous operators Ψ_F for $F \in \Lambda$ built from a limit of finite variable DAHA operators?
- What is the relationship between the $\widetilde{E}_{(\mu|\lambda)}$ and the partially-symmetric Macdonald polynomials of Goodberry [3]?
- Is there a geometric interpretation for the operator Ψ_{p_1}?

References

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