Using Slice Rank & Partition Rank

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FPSAC 2023
UC Davis

July 20, 2023

Overarching Goal:

- Use slice-rank and partition-rank: Given
  - finite set A subset of vector space V
  - A avoids a property $\mathcal{P}$

Goal: **maximize** $|A|$

- Strengthen opportunities to use these using

Partition lattices Π_n

Cap Set

$S \subseteq \mathbb{F}_3^n$ is a **cap set** if S has no 3 collinear points

Goal: **Find $c(n)$: Largest size of a cap set $S \subset \mathbb{F}_3^n$**

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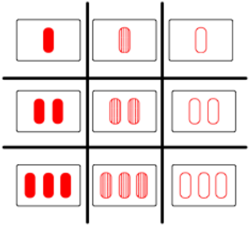
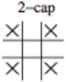


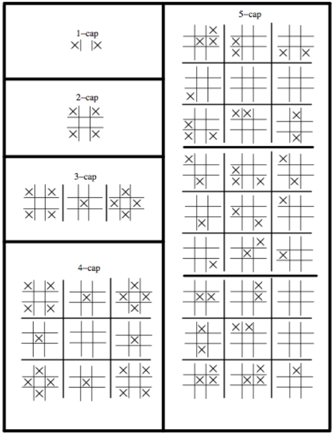
$(0,0)$	$(0,1)$	$(0,2)$
$(1,0)$	$(1,1)$	$(1,2)$
$(2,0)$	$(2,1)$	$(2,2)$



Cap Set

Goal: **Find $c(n)$: Largest size of $S \subseteq \mathbb{F}_3^n$, no 3 points collinear**

$$c(1) = 2 \quad c(2) = 4 \quad c(3) = 9 \quad c(4) = 20 \quad c(5) = 45$$



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Why??

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The following are equivalent:

- $\vec{x}, \vec{y}, \vec{z} \in \mathbb{F}_3^n$ are collinear
- for every i,

$$x_i = y_i = z_i \qquad \text{or} \qquad \{x_i, y_i, z_i\} = \{0, 1, 2\}$$

-

$$\vec{x}, \vec{y}, \vec{z} \text{ form arithmetic progression}$$

Important: Additive Combinatorics & Number Theory

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History:

- Quick: $c(n) \geq 2^n$ (select all points $\{0,1\}^n$)
- Meshulam (1995): $c(n) \leq \frac{2}{n} \cdot 3^n$
- Bateman, Katz (2011): $c(n) < \frac{1}{n^{1+\epsilon}} \cdot 3^n$ JAMS paper

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- Question (Frankl, Graham, Rödl / Alon):

$$c(n) = O(K^n) \text{ for } K < 3?$$

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Yes! 2 Annals Papers!

Theorem (Croot, Lev, Pach / Ellenberg, Gijswijt (2017))

If $S \subset \mathbb{F}_3^n$ does not contain a triple of collinear points then

$$|S| \leq 2.756^n$$

SLICE RANK POLYNOMIAL METHOD

Slice

Definition

A function $f : X^k \to \mathbb{F}$ is a slice if

$$f(x_1, x_2, \dots, x_k) = g(x_i) h(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_k)$$

for some tensors $g : X \to \mathbb{F}$ and $h : X^{k-1} \to \mathbb{F}$.

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- $$f(x_1, x_2, x_3) = x_1x_2 + x_1x_3 + x_2x_3 = x_1(x_2 + x_3) + x_2(x_3)$$

  so slice rank f is 2.

Slice Rank

Theorem (Tao, 2016)

If $f : X^k \to \mathbb{F}$ *is* *diagonal*, *i.e.*

$$f(x_1, x_2, \ldots, x_k) \neq 0 \iff x_1 = x_2 = \cdots = x_k$$

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Example: If $X = \{0, 1, 2\}$ and $f : \{0, 1, 2\}^2 \to \mathbb{F}$ we can write

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so slice-rank$(f) \leq 3$. Theorem above says it is exactly 3.

Strategy

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$$f(\vec{x}, \vec{y}, \vec{z}) = \prod_{i=1}^{n} (x_i + y_i + z_i - 1)(x_i + y_i + z_i - 2).$$

Recall: $\vec{x}, \vec{y}, \vec{z}$ collinear iff

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$$\prod_{i=1}^{n} (x_i + y_i + z_i - 1)(x_i + y_i + z_i - 2) = \sum_{\alpha, \beta, \gamma \in \{0,1,2\}^n} c_{\alpha,\beta,\gamma} \left(\prod_{i=1}^{n} x_i^{\alpha_i} \right) \left(\prod_{i=1}^{n} y_i^{\beta_i} \right) \left(\prod_{i=1}^{n} z_i^{\gamma_i} \right)$$

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This is a sum of slices, the number of which is

$$3 \cdot \left| \left\{ \vec{e} \in \{0,1,2\}^n \ : \ e_1 + e_2 + \cdots + e_n \leq \frac{2n}{3} \right\} \right| \leq 2.756^n.$$

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Many Discoveries!

- Upper bounds on Sunflower-Free Sets: Naslund and Sawin (2017)
- Monochromatic Equilateral Triangles in Unit Distance Graph: Naslund (2019)
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Right Corners (Ge & Shangguan, 2020)

Goal: Maximize $|A|$ for $A \subset \mathbb{F}_q^n$ if A has no right corner: no distinct triple $\vec{x}, \vec{y}, \vec{z} \in \mathbb{F}_q^n$ with

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Attempt: $T : A^3 \to \mathbb{F}_q$

$$T(\vec{x}, \vec{y}, \vec{z}) = \mathbb{1}_{\vec{y} \neq \vec{z}} \cdot ((\vec{z} - \vec{x}) \cdot (\vec{z} - \vec{y}))^{q-1}$$

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Issue: Not Diagonal!

$$T(\vec{x}, \vec{y}, \vec{z}) = \begin{cases} 0 & \text{if } \vec{x} = \vec{y} \text{ or } \vec{x} = \vec{z} \text{ or } \vec{y} = \vec{z} \\ 1 & \text{otherwise} \end{cases}$$

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Fix: Diagonalize

$$T(\vec{x}, \vec{y}, \vec{z}) = 2 \cdot \mathbb{1}_{\vec{x} = \vec{y} = \vec{z}} - \mathbb{1}_{\vec{x} = \vec{y}} - \mathbb{1}_{\vec{x} = \vec{z}} - \mathbb{1}_{\vec{y} = \vec{z}} + 1$$

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$$\mathbb{1}_{\vec{x}=\vec{y}=\vec{z}} = \frac{1}{2}\left(T(\vec{x}, \vec{y}, \vec{z}) + \mathbb{1}_{\vec{x}=\vec{y}} + \mathbb{1}_{\vec{x}=\vec{z}} + \mathbb{1}_{\vec{y}=\vec{z}} - 1\right)$$

$$|A| = \text{slice-rank}(\mathbb{1}_{\vec{x}=\vec{y}=\vec{z}}) \leq \text{slice-rank}(T) + 3.$$

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$$T(\vec{x}, \vec{y}, \vec{z}) = \begin{cases} 0 & \text{if } \vec{x} = \vec{y} \text{ or } \vec{x} = \vec{z} \text{ or } \vec{y} = \vec{z} \\ 1 & \text{otherwise} \end{cases}$$

Fix: Diagonalize

$$T(\vec{x}, \vec{y}, \vec{z}) = 2 \cdot \mathbb{1}_{\vec{x}=\vec{y}=\vec{z}} - \mathbb{1}_{\vec{x}=\vec{y}} - \mathbb{1}_{\vec{x}=\vec{z}} - \mathbb{1}_{\vec{y}=\vec{z}} + 1$$

$$\mathbb{1}_{\vec{x}=\vec{y}=\vec{z}} = \frac{1}{2}\left(T(\vec{x}, \vec{y}, \vec{z}) + \mathbb{1}_{\vec{x}=\vec{y}} + \mathbb{1}_{\vec{x}=\vec{z}} + \mathbb{1}_{\vec{y}=\vec{z}} - 1\right)$$

$$|A| = \text{slice-rank}(\mathbb{1}_{\vec{x}=\vec{y}=\vec{z}}) \leq \text{slice-rank}(T) + 3.$$

Key: Can diagonalize T since it is constant on partitions.

k-Right Corners (Naslund 2020)

Definition (Naslund 2020)

Distinct $x_1, \ldots, x_k, x_{k+1} \in \mathbb{F}_q^n$ form a k-right corner if they are distinct and

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$$F_k = \prod_{j<\ell\leq k} \left(1 - \langle x_j - x_{k+1}, x_\ell - x_{k+1}\rangle^{q-1}\right).$$

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Not Diagonal! Example:

$$x_1 - x_{k+1} \perp x_2 - x_{k+1},\ \ x_2 - x_{k+1} \text{ self orthogonal, } x_i = x_2 \text{ for } 2 \leq i \leq k$$

General Framework

- $A \subset \mathbb{F}_q^n$, no distinct k-tuple has property $\mathcal{P}$
- "Easy" To Create $F_k : (\mathbb{F}_q^n)^k \to \mathbb{F}$ with

$$F_k(x_1, \ldots, x_k) = \begin{cases} c_1 & \text{if } x_1, \ldots, x_k \text{ (not necessarily distinct!) satisfies } \mathcal{P} \\ c_2 \ (\neq 0) & \text{if } x_1 = x_2 = \cdots = x_k \\ 0 & \text{otherwise} \end{cases}$$

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- Upper bounds on Sunflower-Free Sets: Naslund and Sawin (2017)
- Monochromatic Equilateral Triangles in Unit Distance Graph: Naslund (2019)
- Erdős-Ginzburg-Ziv Constant: Naslund (2019)
- Sets avoiding Right Corners: Ge, Shangguan (2020)
- Solutions to Linear Systems in Finite Fields: Sauermann (2021)

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- $H_3(x_1, x_2, x_3) = 1 - \mathbb{1}_{x_1 = x_2} - \mathbb{1}_{x_1 = x_3} - \mathbb{1}_{x_2 = x_3}$
- Möbius Inversion on Π_3

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- $|A|$ bounded above by slice-rank$(F_3 \cdot H_3)$

$$\text{slice-rank}(F_3) + \text{slice-rank}(\mathbb{1}_{x_1=x_2} \cdot F_3) + \text{slice-rank}(\mathbb{1}_{x_1=x_3} \cdot F_3) + \text{slice-rank}(\mathbb{1}_{x_2=x_3} \cdot F_3)$$

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- $$\begin{aligned} H_4(x_1, x_2, x_3, x_4) = & 1 - \mathbb{1}_{x_1=x_2} - \mathbb{1}_{x_1=x_3} - \mathbb{1}_{x_2=x_3} - \mathbb{1}_{x_1=x_4} - \mathbb{1}_{x_2=x_4} - \mathbb{1}_{x_3=x_4} \\ & + 2 \cdot \mathbb{1}_{x_1=x_2=x_3} + 2 \cdot \mathbb{1}_{x_1=x_2=x_4} + 2 \cdot \mathbb{1}_{x_1=x_3=x_4} + 2 \cdot \mathbb{1}_{x_2=x_3=x_4} \\ & + \mathbb{1}_{x_1=x_2} \cdot \mathbb{1}_{x_3=x_4} + \mathbb{1}_{x_1=x_3} \cdot \mathbb{1}_{x_2=x_4} + \mathbb{1}_{x_1=x_4} \cdot \mathbb{1}_{x_2=x_3} \\ & = 6 \cdot \mathbb{1}_{x_1=x_2=x_3=x_4} \text{ on } A^4 \text{ (Möbius inversion on } \Pi_4) \end{aligned}$$

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  ????????

Partition Rank Method (Naslund 2020)

PARTITION RANK POLYNOMIAL METHOD

Overview

Partition Rank Method

Strategy:

- Step 1: $X \subset \mathbb{F}^n$ avoids a property, find a diagonal tensor

$$f : X^k \to \mathbb{F}$$

- Step 2: Find upper bound on slice-rank(f) partition-rank(f), then:

$$|X| = \underbrace{\text{partition-rank}(f)}_{\leq \text{ slice-rank}(f)} \leq \text{upper bound}$$

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Definition

A function $f : X^k \to \mathbb{F}$ has partition rank 1 if there is a set partition π of $\{1, 2, \ldots, k\}$ with blocks $\pi_1, \pi_2, \ldots, \pi_\ell$ so that

$$f(x_1, x_2, \ldots, x_k) = \prod_{i=1}^{\ell} f_{\pi_i}$$

where f_{π_i} is a function in the variables $\{x_j \ : \ j \in \pi_i\}$

Example

If $f : X^4 \to \mathbb{F}$ with

$$f(x_1, x_2, x_3, x_4) = \begin{cases} 1 & \text{if } x_1 = x_2 \text{ and } x_3 = x_4 \\ 0 & \text{otherwise} \end{cases}$$

then $f(x_1, x_2, x_3, x_4) = \mathbb{1}_{x_1 = x_2} \cdot \mathbb{1}_{x_3 = x_4}$ so it has partition rank 1. Its slice rank is ???

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Theorem (Naslund 2019)

If $f : X^k \to \mathbb{F}$ is <u>diagonal</u>, i.e.

$$f(x_1, x_2, \ldots, x_k) \neq 0 \iff x_1 = x_2 = \cdots = x_k$$

then

$$\textit{partition-rank}(f) = |X|.$$

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- Unifies approaches in papers of Naslund:

$$\text{distinctness indicator} \implies \text{partition indicator}$$

- Finite field analogue of problem of Erdös
- Generalizes work of Burscis, Matolcsi, Pach, Schrettner

THANKS