Fine Polyhedral Adjunction Theory
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Joint work with Christian Haase

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Some figures taken from Andreas Paffenholz.

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Adjunction Theory
Adjunction Theory

Polyhedral Adjunction Theory
Overview

Adjunction Theory

\[ \downarrow \]

Polyhedral Adjunction Theory

\[ \downarrow \]

Fine Polyhedral Adjunction Theory
Overview

Adjunction Theory

Polyhedral Adjunction Theory

Fine Polyhedral Adjunction Theory

Two main results
Adjunction Theory
Studies Polarized Varieties:

\((X,L)\)

- **X**: Normal Projective Algebraic Variety
- **L**: Ample line bundle on **X**
Adjunction Theory: Toric Geometry

Studies Polarized Varieties:

\[(X,L)\]

- \(X\): Normal Projective Algebraic Variety
- \(L\): Ample line bundle on \(X\)

Precursor of minimal model program: Classification of varieties.

- **Miles Reid et al. (1985):** Young Person’s Guide to Canonical Singularities.
Studies Polarized Varieties:

\[(X, L)\]

- \(X\): Normal Projective Algebraic Variety
- \(L\): Ample line bundle on \(X\)

Precursor of minimal model program: Classification of varieties.


Classification of toric varieties via duality:

Polyhedral Adjunction Theory
Combinatorial Counterpart of Adjunction Theory

Using dictionary between polytopes and toric varieties.
Combinatorial Counterpart of Adjunction Theory

Using dictionary between polytopes and toric varieties.

Studies lattice polytopes.

- Polytopes whose vertices have integer coordinates.
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Motivation: Fine Version used to construct Minimal Models.

- **Batyrev (Appeared 2020, Published July 2023)**: Canonical Models of Toric Hypersurfaces.
First Definitions

Definition

Let $P \subseteq \mathbb{R}^n$ a rational polytope of dimension $n$, i.e.,

$$P = \{ x \in \mathbb{R}^n | \langle a_i, x \rangle \geq b_i, i = 1, \ldots, m \}$$

for $a_i$ primitive rows of an integer matrix $A$ and $b \in \mathbb{Q}^m$.

Each inequality $\langle a_i, \cdot \rangle \geq b_i$ defines a facet $F_i$ of $P$. 
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Definition
For any $s \geq 0$ the adjoint polytope is

$$P^{(s)} = \{ x \in \mathbb{R}^n | Ax \geq b + s1 \}$$

with $1 = (1, \ldots, 1)^T$. 
First Definitions

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For any \( s \geq 0 \) the **adjoint polytope** is

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P^{(s)} = \{ x \in \mathbb{R}^n | Ax \geq b + s \mathbf{1} \}
\]

with \( \mathbf{1} = (1, ..., 1)^T \).

**Polyhedral Adjunction Theory:** Study of \( P^{(s)} \).
Figure: Original Polytope $P$. 
Example

Figure: Adjoint Polytope $P(s)$ for $s = 3/2$. 

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Figure: Adjoint Polytope $P^{(s)}$ for $s = 2$. 
Figure: Adjoint Polytope $P^{(s)}$ for $s = 5/2$. 
Figure: Adjoint Polytope $P^{(s)}$ for $s = 3$. 
A First Invariant

Definition

The $\mathbb{Q}$-codegree of $P$ is

$$\mu(P) := \left( \sup \{ s > 0 \mid P^{(s)} \neq \emptyset \} \right)^{-1}.$$ 

The core of $P$ is

$$\text{core}(P) := P^{(1/\mu(P))}.$$
A First Invariant

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In our example:

$$\mu(P) = \frac{1}{3}, \quad \text{core}(P) = P^{(1/\mu(P))}.$$
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In our example:

$$
\mu(P) = \frac{1}{3}, \quad \text{core}(P) = P^{(3)}.
$$
Definition

The **codegree** of $P$ is

$$\text{cd}(P) := \min\{k \in \mathbb{Z}_\geq 1 \mid \text{int}(kP) \cap \mathbb{Z}^n \neq \emptyset\}.$$
A Second Invariant

### Definition

The **codegree** of $P$ is

$$\text{cd}(P) := \min\{k \in \mathbb{Z}_{\geq 1} | \text{int}(kP) \cap \mathbb{Z}^n \neq \emptyset\}.$$ 

For a lattice polytope $P$,

$$\text{int}(P) \cap \mathbb{Z}^n = P^{(1)} \cap \mathbb{Z}^n.$$ 

Hence,

$$\mu(P) \leq \text{cd}(P) \leq n + 1.$$
Cayley Polytopes interesting from the polytope point of view:

- Having Cayley Decomposition is a strong structural statement. Cayley Polytopes have projections onto unimodular simplexes.

Cayley Polytopes interesting from the toric geometry point of view:

- Their corresponding polarized toric variety is birationally fibered.
Cayley Polytopes interesting from the *polytope* point of view:
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Asking the reverse question:
- Dimension of a polytope gives a bound for $\mu(P)$.
  Can we derive from $\mu(P)$ a bound for the dimension?
Cayley Polytopes

Definition

The **lattice width** of a polytope $P$ is

$$w_P = \min \{ w_P(u) \mid u \text{ is a non-zero integer linear form} \}$$

where

$$w_P(u) := \max_{x \in P} \langle u, x \rangle - \min_{x \in P} \langle u, x \rangle.$$
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**Definition**

A **Cayley Polytope** is a lattice polytope of lattice width 1.
Cayley Sums

Definition

For lattice polytopes \( P_0, \ldots, P_t \) in \( \mathbb{R}^k \), the \textbf{Cayley sum} is

\[
P_0 \star \cdots \star P_t := \text{conv}(P_0 \times 0) \cup (P_1 \times e_1) \cup \cdots \cup (P_t \times e_t)
\]

for the standard basis \( e_1, \ldots, e_t \) of \( \mathbb{R}^t \).

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P_0 \star \cdots \star P_t \subseteq \mathbb{R}^k \times \mathbb{R}^t.
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Figure: Cayley Sum of Polytopes.
Cayley Sums

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for the standard basis $e_1, \ldots, e_t$ of $\mathbb{R}^t$.

$$P_0 \star \cdots \star P_t \subseteq \mathbb{R}^k \times \mathbb{R}^t.$$
The Decomposition Theorem

\[ d(P) = \begin{cases} 
2(n - \lceil \mu(P) \rceil) & \text{if } \mu(P) \in \mathbb{N} \\
2(n - \mu(P)) + 1 & \text{if } \mu(P) \notin \mathbb{N}
\end{cases} \]

**Theorem (Di Rocco, Haase, Nill, Paffenholz)**

Let \( P \) an \( n \)-dimensional lattice polytope with \( P \neq \Delta_n \). If \( n > d(P) \), then \( P \) is a Cayley sum of lattice polytopes in \( \mathbb{R}^m \) with \( m \leq d(P) \).
Characterization requires two conditions:

1. **Bounded $\mathbb{Q}$-Codegree:** For $\varepsilon > 0$, $\mu(P) \geq \varepsilon$.

2. **$\alpha$-canonicity:** The normal fan of $P$ is $\alpha$-canonical.
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   Any lattice point $p = \sum_i \lambda_i a_i$ on a cone of $\mathcal{N}(P)$ generated by $a_i$ has $\sum_i \lambda_i \geq \alpha$. 

**Theorem (Paffenholz)**

Let $n \in \mathbb{N}$ and $\alpha, \varepsilon > 0$ be given. Then 
\[ \{ \mu(P) \mid P \in S_{\text{can}}(n, \varepsilon) \} \] 

is finite.

$S_{\text{can}}(n, \varepsilon)$: Set of $n$-dimensional lattice polytopes with $\mu(P) \geq \varepsilon$ and $\alpha$-canonical normal fan.
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Can we improve this?

Fine Polyhedral Adjunction Theory
Fine Polyhedral Adjunction Theory

**Fine (1983):** Resolution and completion of algebraic varieties.
**Idea:** Take adjoint polytopes with respect to *all* valid inequalities for $P$. 
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*Figure:* Valid Inequalities for a polytope $P$. 

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New Definitions

Definition

For $s > 0$, the **Fine adjoint polytope** is

$$P^F(s) := \{x \in \mathbb{R}^n | d^F(x) \geq s\}$$
New Definitions

**Definition**

The **Fine $\mathbb{Q}$-codegree** of a rational polytope $P$ is

$$\mu^F(P) := (\sup\{s > 0|P^F(s) \neq \emptyset\})^{-1},$$

and the **Fine core** of $P$ is

$$\text{core}^F(P) := P^F(1/\mu^F(P)).$$
Adjoints vs. Fine Adjoints

Figure: Original Adjoints

Figure: Fine Adjoints
Adjoints vs. Fine Adjoints

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Adjoints vs. Fine Adjoints

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Figure: Fine Adjoints
Main Result 1: Fine Decomposition Theorem

Define

$$d^F(P) := \begin{cases} 2(n - \lfloor \mu^F(P) \rfloor), & \text{if } \mu^F(P) \notin \mathbb{N} \\ 2(n - \mu^F(P)) + 1, & \text{if } \mu^F(P) \in \mathbb{N} \end{cases}$$

In general:

$$\mu(P) \leq \mu^F(P)$$

Theorem (G., Haase)

Let $P$ an $n$-dimensional lattice polytope with $P \not= \Delta_n$. If $n > d^F(P)$, then $P$ is a Cayley sum of lattice polytopes in $\mathbb{R}^m$ with $m \leq d^F(P)$.

This theorem is slightly stronger.
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This theorem is slightly stronger.
Characterization now requires only **one** condition:

- **Bounded Fine $\mathbb{Q}$-codegree:** For $\varepsilon > 0$, $\mu^F(P) \geq \varepsilon$. 
Main Result 2: Fine $\mathbb{Q}$-Codegree Spectrum

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**The $\alpha$-canonicity assumption on the normal fan can be dropped.**

Moving in by all valid inequalities heavily restricts the value that $\mu^F(P)$ can take.
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Let $n \in \mathbb{N}$ and $\varepsilon > 0$ be given. Then

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is finite.

$S^F(n, \varepsilon)$: Set of $n$-dimensional lattice polytopes with $\mu^F(P) \geq \varepsilon$. 
Start with original invariants and adjoints.

Redefine them by adding the word “Fine” to them.

Invariants play a role in the Cayley structure of polytopes and their $\mathbb{Q}$-codegree values.

We obtain stronger results and easier proofs about Cayley structures, projections and Fine $\mathbb{Q}$-codegree values.

"Fine theory is nicer than the old theory."

- Christian Haase (2023)
Summary

Start with original invariants and adjoints.

Redefine them by adding the word “Fine” to them.

Invariants play a role in the Cayley structure of polytopes and their $\mathbb{Q}$-codegree values.

We obtain stronger results and easier proofs about Cayley structures, projections and Fine $\mathbb{Q}$-codegree values.

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Open Questions

- **Bounds:** Are there better bounds for the dimension of polytopes with Cayley structure? Studied by [Dickenstein, Nill], [Di Rocco, Haase, Nill, Paffenholz],...

- **Inverse game:** How to translate Fine invariants into toric geometry?

- **Further results:** What other notions from Polyhedral Adjunction Theory can we translate to the Fine case?

- ...

A Third Invariant

Definition

The **nef value** of $P$ is

$$\tau(P) := (\sup\{s > 0|N(P^{(s)}) = N(P)\})^{-1}.$$
Pair $(X, L)$:
- $X$ a projective variety,
- $L$ an ample line bundle on $X$.

Adjunction Theory studies adjoint linear systems $L + cK_X$.

- **Nef-value:** $\tau := (\sup\{c \in \mathbb{R} | L + cK_X \text{ is ample}\})^{-1}$
- **Q-codegree:** $\mu := (\sup\{c \in \mathbb{R} | L + cK_X \text{ is big}\})^{-1}$

$\text{Ample} \Rightarrow \text{Big}: \mu \leq \tau$. 

$\mathbb{Q}$-codegree and nef value in adjunction theory

Figure: Adjunction theory point of view
Before: Under natural projection $\pi_P$, if $Q = \pi_P(P)$, then

$$\mu(P) \leq \mu(Q).$$

**Theorem**

The image $Q := \pi_P(P)$ of the natural projection of $P$ is a rational polytope satisfying

$$\mu^F(Q) = \mu^F(P).$$

Moreover, then $\text{core}^F(Q)$ is the point $\pi_P(\text{core}^F(P))$. 
Natural Projection in the Fine Case

Figure: Behaviour of the Fine core.

\[ P = \text{conv} \begin{bmatrix} 0 & 2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 & h & h \end{bmatrix} \]