Fine Polyhedral Adjunction Theory

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Joint work with Christian Haase

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Some figures taken from Andreas Paffenholz.

FPSAC 2023

Adjunction Theory

Adjunction Theory

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Polyhedral Adjunction Theory

Adjunction Theory

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Polyhedral Adjunction Theory

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Fine Polyhedral Adjunction Theory

Adjunction Theory

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Polyhedral Adjunction Theory

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Fine Polyhedral Adjunction Theory

Two main results

Adjunction Theory

Studies Polarized Varieties:

$$\boxed{(X,L)}$$

- X: Normal Projective Algebraic Variety
- L: Ample line bundle on X

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Precursor of minimal model program: Classification of varieties.

- **Miles Reid et al. (1985):** Young Person's Guide to Canonical Singularities.

Adjunction Theory: Toric Geometry

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- **Miles Reid et al. (1985):** Young Person's Guide to Canonical Singularities.

Classification of toric varieties via duality:

- **Dickenstein, Di Rocco, Piene (2014):** Higher Order Duality and Toric Embeddings.

Polyhedral Adjunction Theory

Combinatorial Counterpart of Adjunction Theory

Using dictionary between polytopes and toric varieties.

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Studies lattice polytopes.

- Polytopes whose vertices have integer coordinates.
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- **Di Rocco, Haase, Nill, Paffenholz (2014):** Polyhedral Adjunction Theory.

Motivation: Fine Version used to construct Minimal Models.

- **Batyrev (Appeared 2020, Published July 2023):** Canonical Models of Toric Hypersurfaces.

First Definitions

Definition

Let $P \subseteq \mathbb{R}^n$ a rational polytope of dimension n, i.e.,

$$P = \{x \in \mathbb{R}^n | \langle a_i, x \rangle \geq b_i, i = 1, ..., m\}$$

for a_i primitive rows of an integer matrix A and $b \in \mathbb{Q}^m$.

Each inequality $\langle a_i, \cdot \rangle \geq b_i$ defines a facet F_i of P.

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For any $s \geq 0$ the **adjoint polytope** is

$$P^{(s)} = \{x \in \mathbb{R}^n | Ax \geq b + s\mathbb{1}\}$$

with $\mathbb{1} = (1, ..., 1)^T$.

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Polyhedral Adjunction Theory: Study of $P^{(s)}$.

Example

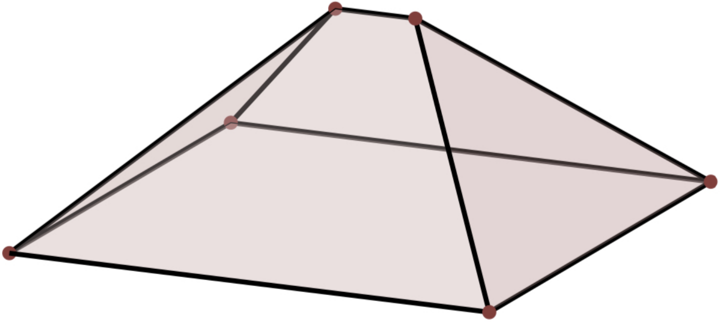

Figure: Original Polytope P.

Example

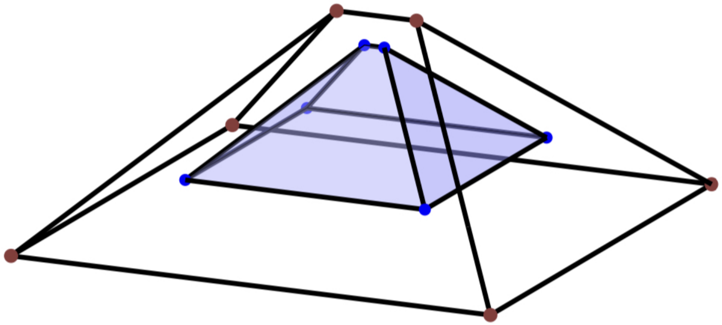

Figure: Adjoint Polytope $P^{(s)}$ for $s = 3/2$.

Example

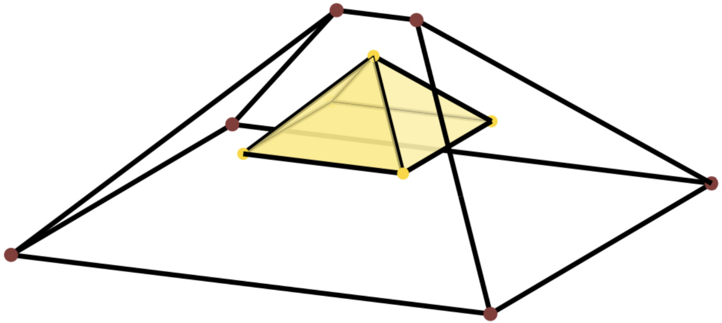

Figure: Adjoint Polytope $P^{(s)}$ for $s = 2$.

Example

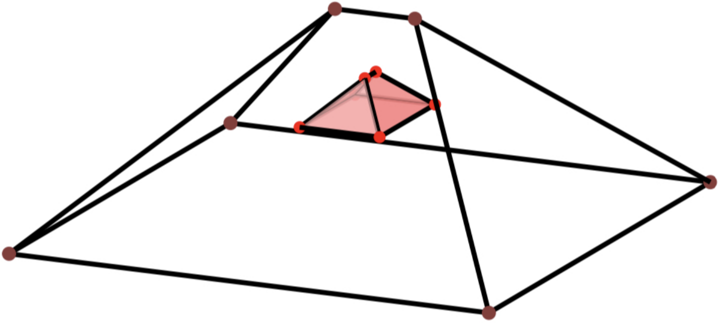

Figure: Adjoint Polytope $P^{(s)}$ for $s = 5/2$.

Example

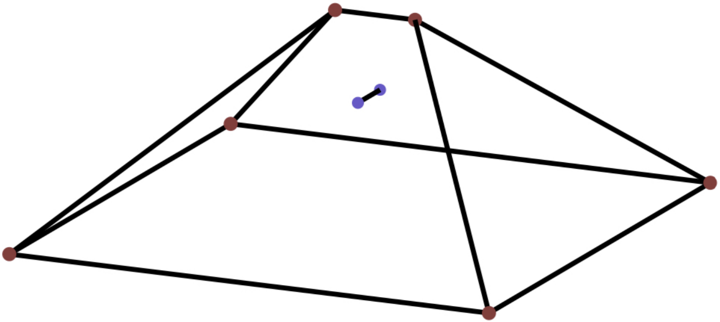

Figure: Adjoint Polytope $P^{(s)}$ for $s = 3$.

A First Invariant

Definition

The $\mathbb{Q}$**-codegree** of P is

$$\mu(P) := (\sup\{s > 0 | P^{(s)} \neq \emptyset\})^{-1}.$$

The **core** of P is

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In our example:
$$\mu(P) = \frac{1}{3}, \qquad \text{core}(P) = P^{(3)}.$$

A Second Invariant

Definition

The **codegree** of P is

$$\mathrm{cd}(P) := \min\{k \in \mathbb{Z}_{\geq 1} \,|\, \mathrm{int}(kP) \cap \mathbb{Z}^n \neq \emptyset\}.$$

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For a lattice polytope P,

$$\mathrm{int}(P) \cap \mathbb{Z}^n = P^{(1)} \cap \mathbb{Z}^n.$$

Hence,

$$\mu(P) \leq \mathrm{cd}(P) \leq n + 1.$$

Cayley Polytopes interesting from the *polytope* point of view:

- Having Cayley Decomposition is a strong structural statement. Cayley Polytopes have projections onto unimodular simplexes.

Cayley Polytopes interesting from the *toric geometry* point of view:

- Their corresponding polarized toric variety is birationally fibered.

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Asking the reverse question:

Dimension of a polytope gives a bound for $\mu(P)$.
Can we derive from $\mu(P)$ a bound for the dimension?

Definition

The **lattice width** of a polytope P is

$$w_P = \min\{w_P(u) | u \text{ is a non-zero integer linear form}\}$$

where

$$w_P(u) := \max_{x \in P} \langle u, x \rangle - \min_{x \in P} \langle u, x \rangle.$$

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Definition

A **Cayley Polytope** is a lattice polytope of lattice width 1.

Cayley Sums

Definition

For lattice polytopes $P_0, ..., P_t$ in $\mathbb{R}^k$, the **Cayley sum** is

$$P_0 \star \cdots \star P_t := \operatorname{conv}(P_0 \times 0) \cup (P_1 \times e_1) \cup \cdots \cup (P_t \times e_t)$$

for the standard basis $e_1, ..., e_t$ of $\mathbb{R}^t$.

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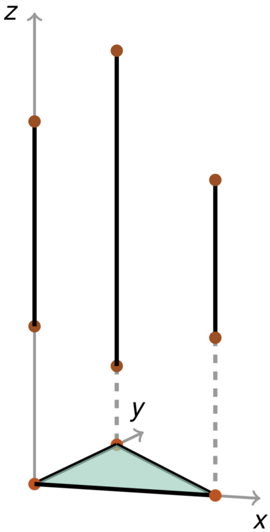

Figure: Cayley Sum of Polytopes.

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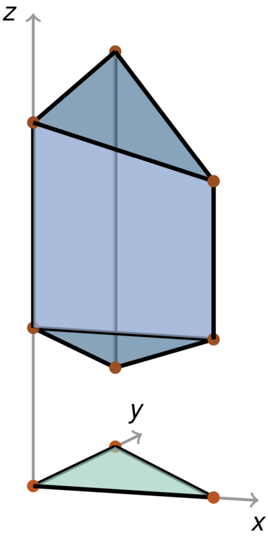

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The Decomposition Theorem

$$d(P) = \begin{cases} 2(n - \lfloor \mu(P) \rfloor) & \text{if } \mu(P) \notin \mathbb{N} \\ 2(n - \mu(P)) + 1 & \text{if } \mu(P) \in \mathbb{N} \end{cases}$$

Theorem (Di Rocco, Haase, Nill, Paffenholz)

Let P an n-dimensional lattice polytope with $P \ncong \Delta_n$. If $n > d(P)$, then P is a Cayley sum of lattice polytopes in $\mathbb{R}^m$ with $m \leq d(P)$.

$\mathbb{Q}$-Codegree Spectrum: What values can $\mu(P)$ take?

Characterization requires **two** conditions:

1. **Bounded $\mathbb{Q}$-Codegree:** For $\varepsilon > 0$, $\mu(P) \geq \varepsilon$.
2. **α-canonicity:** The normal fan of P is α-canonical.

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Theorem (Paffenholz)

Let $n \in \mathbb{N}$ and $\alpha, \varepsilon > 0$ be given. Then

$$\{\mu(P) | P \in \mathcal{S}_\alpha^{can}(n, \varepsilon)\}$$

is finite.

$\mathcal{S}_\alpha^{can}(n, \varepsilon)$: Set of n-dimensional lattice polytopes with $\mu(P) \geq \varepsilon$ and α-canonical normal fan.

Fine Polyhedral Adjunction Theory

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Fine (1983): Resolution and completion of algebraic varieties.

Idea: Take adjoint polytopes with respect to *all* valid inequalities for P.

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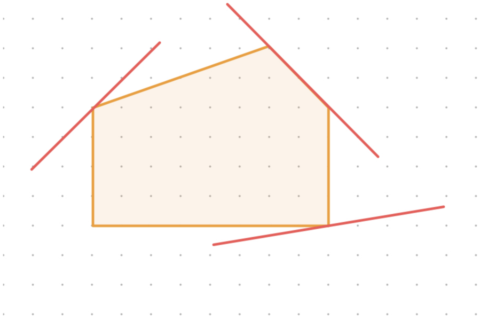

Figure: Valid Inequalities for a polytope P.

New Definitions

Definition

For $s > 0$, the **Fine adjoint polytope** is

$$P^{F(s)} := \{x \in \mathbb{R}^n | d^F(x) \geq s\}$$

New Definitions

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The **Fine $\mathbb{Q}$-codegree** of a rational polytope P is

$$\mu^F(P) := (\sup\{s > 0 | P^{F(s)} \neq \emptyset\})^{-1},$$

and the **Fine core** of P is

$$\mathrm{core}^F(P) := P^{F(1/\mu^F(P))}.$$

Adjoints vs. Fine Adjoints

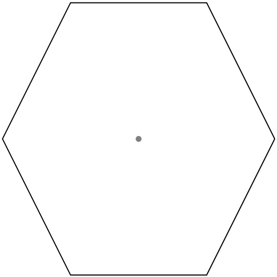
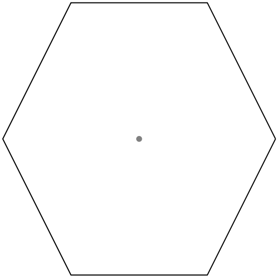


Figure: Original Adjoints



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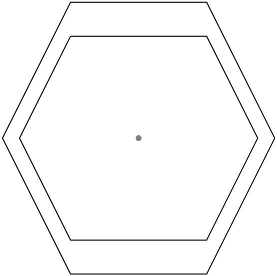

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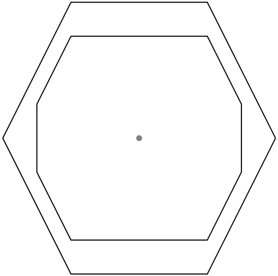

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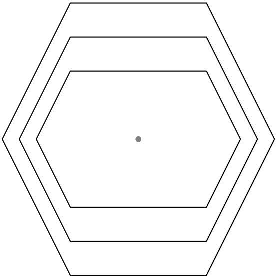

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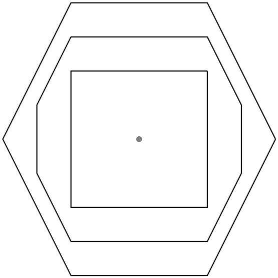

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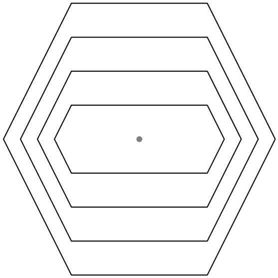

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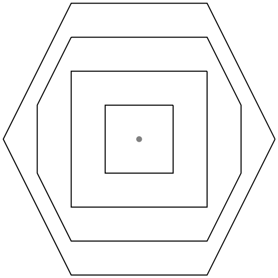

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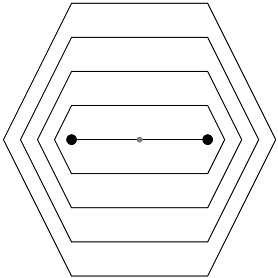

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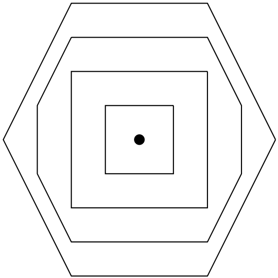

Figure: Fine Adjoints

Main Result 1: Fine Decomposition Theorem

Define

$$d^F(P) := \begin{cases} 2(n - \lfloor \mu^F(P) \rfloor), \text{ if } \mu^F(P) \notin \mathbb{N} \\ 2(n - \mu^F(P)) + 1, \text{ if } \mu^F(P) \in \mathbb{N} \end{cases}$$

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Theorem (G., Haase)

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This theorem is slightly **stronger**.

Main Result 2: Fine $\mathbb{Q}$-Codegree Spectrum

Characterization now requires only **one** condition:

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$\mathcal{S}^F(n, \varepsilon)$: Set of n-dimensional lattice polytopes with $\mu^F(P) \geq \varepsilon$.

Start with original invariants and adjoints.

$\downarrow$

Redefine them by adding the word "Fine" to them.

Invariants play a role in the Cayley structure of polytopes and their $\mathbb{Q}$-codegree values.

$\downarrow$

We obtain stronger results and easier proofs about Cayley structures, projections and Fine $\mathbb{Q}$-codegree values.

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“Fine theory is nicer than the old theory”.

-Christian Haase (2023)-

- **Bounds:** Are there better bounds for the dimension of polytopes with Cayley structure?
  Studied by [Dickenstein, Nill], [Di Rocco, Haase, Nill, Paffenholz],...

- **Inverse game:** How to translate Fine invariants into toric geometry?

- **Further results:** What other notions from Polyhedral Adjunction Theory can we translate to the Fine case?

- . . .

References

- Sofía Garzón Mora and Christian Haase, *Fine Polyhedral Adjunction Theory*, arXiv:2302.04074, 2023.

1. Andreas Paffenholz, *Polyhedral Adjunction Theory*, Nov. 2012, https://polymake.org/polytopes/paffenholz/data/preprints/1211-sydney-talk.pdf
2. Sandra Di Rocco, Christian Haase, Benjamin Nill, and Andreas Paffenholz. *Polyhedral adjunction theory.* Algebra & Number Theory 7, no. 10, 2417-2446, 2014.
3. Andreas Paffenholz. *Finiteness of the polyhedral $\mathbb{Q}$-codegree spectrum.* Proceedings of the American Mathematical Society 143, no. 11, 4863-4873, 2015.
4. Miles Reid et al. Young person's guide to canonical singularities. Algebraic geometry, Bowdoin, 46:345– 414, 1985.

A Third Invariant

Definition

The **nef value** of P is

$$\tau(P) := (\sup\{s > 0 | \mathcal{N}(P^{(s)}) = \mathcal{N}(P)\})^{-1}.$$

Pair (X, L):

- X a projective variety,
- L an ample line bundle on X.

Adjunction Theory studies adjoint linear systems $L + cK_X$.

- **Nef-value:** $\tau := (\sup\{c \in \mathbb{R} | L + cK_X \text{ is ample}\})^{-1}$
- $\mathbb{Q}$**-codegree:** $\mu := (\sup\{c \in \mathbb{R} | L + cK_X \text{ is big}\})^{-1}$

Ample $\Rightarrow$ Big: $\mu \leq \tau$.

$\mathbb{Q}$-codegree and nef value in adjunction theory

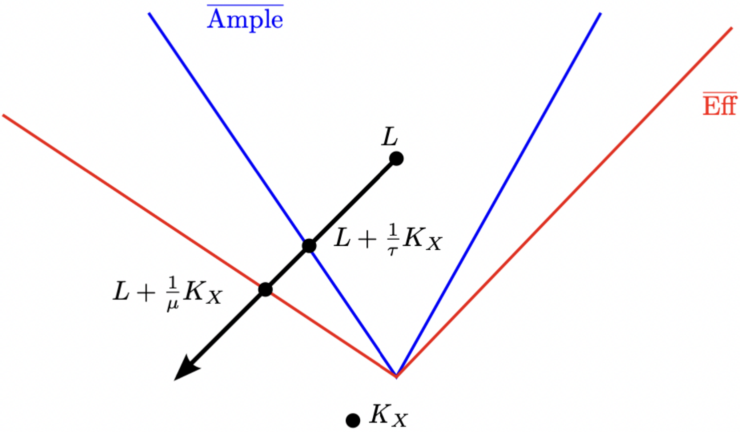


Figure: Adjunction theory point of view

Natural Projection in the Fine case

Before: Under natural projection π_P, if $Q = \pi_P(P)$, then

$$\mu(P) \leq \mu(Q).$$

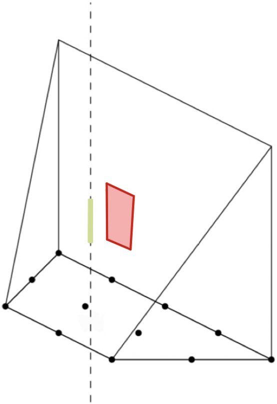
Theorem

The image $Q := \pi_P(P)$ of the natural projection of P is a rational polytope satisfying

$$\mu^F(Q) = \mu^F(P).$$

Moreover, then $\mathrm{core}^F(Q)$ is the point $\pi_P(\mathrm{core}^F(P))$.

Natural Projection in the Fine Case



$$P = \mathrm{conv}\begin{bmatrix} 0 & 2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 & h & h \end{bmatrix}$$

Figure: Behaviour of the Fine core.