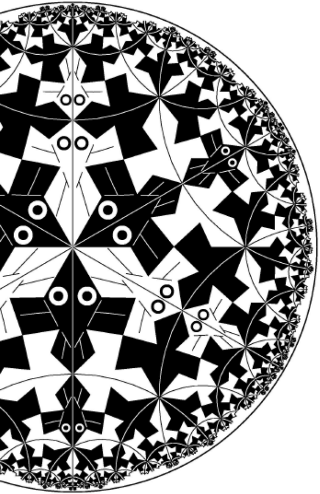
Compact Hyperbolic Coxeter d-Polytopes with $d + 4$ Facets and Related Dimension Bounds



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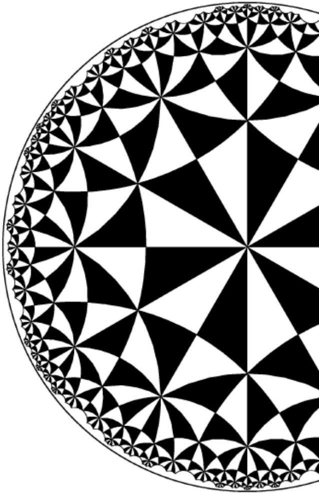
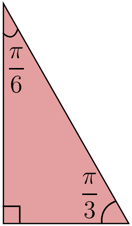
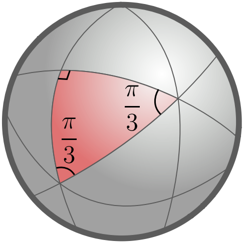
Based on a master's thesis advised by Pavel Tumarkin

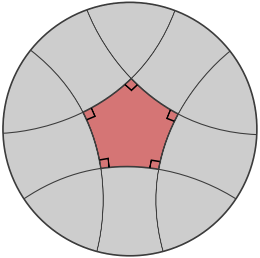


Coxeter Polytopes

A polytope P in Euclidean, spherical, or hyperbolic space is called a *Coxeter polytope* if all of its dihedral angles (angles between facets) are equal to π/m for some $m \in \mathbb{N}$.







- Their study was popularized by Coxeter's classifications in Euclidean and spherical space in 1934.
- These polytopes are heavily involved in the study of minimal volume orbifolds [see work of Meyeroff, Kellerhals, and Hild].

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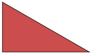

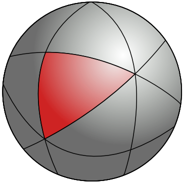

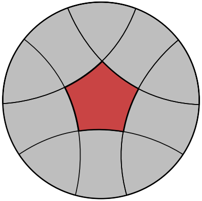

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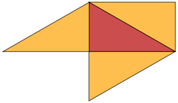

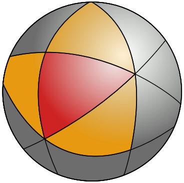

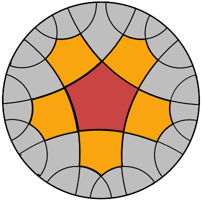

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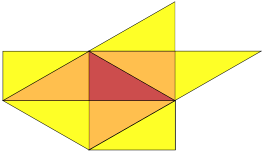

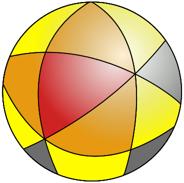

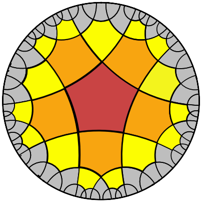

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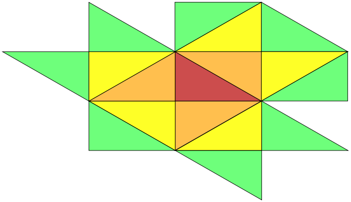

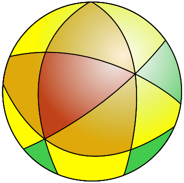

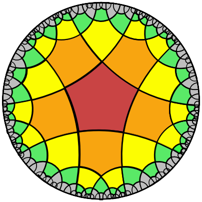

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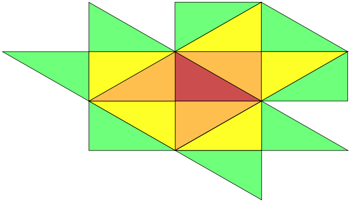
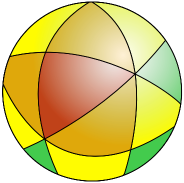
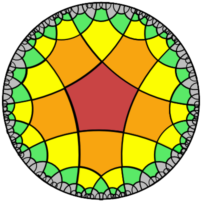




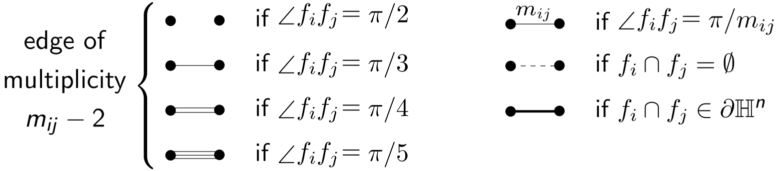


Each reflection group is a Coxeter group $\left\langle r_{f_i} : r_{f_i}^2 = e, (r_{f_i} r_{f_j})^{m_{ij}} \right\rangle$ where f_i denote the facets and $\angle f_i f_j = \pi / m_{ij}$ are the dihedral angles.

Coxeter Diagrams

We represent Coxeter polytopes by *Coxeter diagrams*, which encode the information of how the facets intersect.

Each facet f_i is represented by a node, and the nodes are connected by edges as follows:



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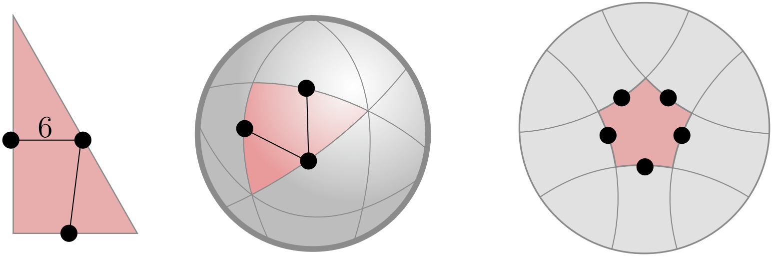
edge of multiplicity $m_{ij} - 2$:

- if $\angle f_i f_j = \pi/2$
- if $\angle f_i f_j = \pi/3$
- if $\angle f_i f_j = \pi/4$
- if $\angle f_i f_j = \pi/5$

m_{ij} if $\angle f_i f_j = \pi/m_{ij}$

if $f_i \cap f_j = \emptyset$

if $f_i \cap f_j \in \partial\mathbb{H}^n$



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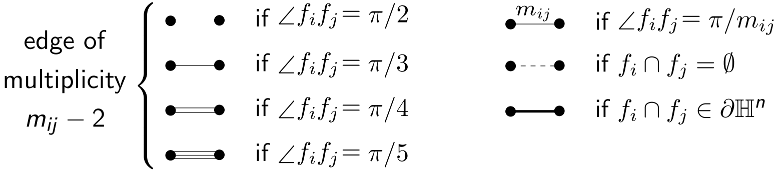
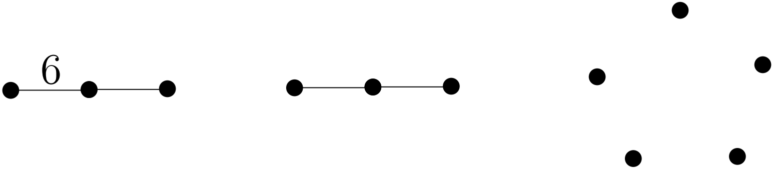
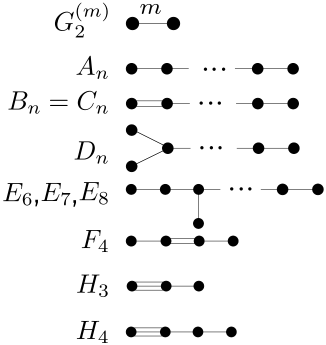


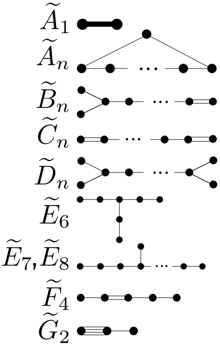


Euclidean and Spherical Coxeter Polytopes

Spherical Coxeter polytopes have Coxeter diagrams that are disjoint unions of *elliptic* diagrams

Euclidean Coxeter polytopes have Coxeter diagrams that are disjoint unions of *parabolic* diagrams

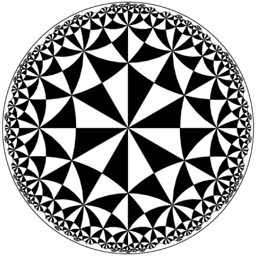




These are related to finite and affine Dynkin diagrams, respectively.

Compact Hyperbolic Polytopes

- A vertex on the boundary of hyperbolic space is called an *ideal vertex*.
- A *compact* polytope in hyperbolic space is a polytope with no ideal vertices. We will work entirely with compact polytopes.
- Polytopes with ideal vertices can still have finite volume, and finite-volume Coxeter polytopes are another area of much interest.



compact

non-compact

Hyperbolic Classification History

By Dimension d

- $d = 2$: all n-gons with angle sum $< \pi(n-2)$, by Poincaré (1882)
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- $n = d + 1$: exist when $1 \leq d \leq 4$, classified by Lannér (1950)
- $n = d + 2$: exist when $2 \leq d \leq 5$, classified by Kaplinskaya (1974) and Esselmann (1996)
- $n = d + 3$: exist when $2 \leq d \leq 6$ and $d = 8$, classified by Tumarkin (2007) using dimension bounds from Esselmann (1994)

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- $n \geq d + 5$: only sporadic families are known

For more info, see Anna Felikson's *Hyperbolic Coxeter Polytopes* webpage.

Classification Strategy

For d-polytopes with $d + 4$ facets, we have

- infinitely many polytopes for $d = 2$ and 3 (Poincaré and Andreev),
- one polytope in dimension 7 (Felikson and Tumarkin), and
- no polytopes in dimension greater than 7.

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We classify the polytopes in dimensions 4 and 5 by successively applying

- geometric restrictions (via Gale diagrams),
- graph theoretic restrictions (via Lannér subdiagrams), and
- algebraic restrictions (via Gram matrices),

until the problem is reduced to a finite computation feasible for a computer.

Geometric Lens: Gale Diagrams

Let N be the normal vectors to the $n = d + k$ facets of a d-polytope P.

- By taking the dual of N as an oriented matroid in $\mathbb{R}^{d+k+1}$, we obtain an oriented matroid in $\mathbb{R}^{k-1}$.
- The resulting set of points in $\mathbb{R}^{k-1}$ is the *Gale diagram* $G(P) = \{p_i\}_{i \in [n]}$ of P. By normalizing, we can assume $G(P) \subseteq S^{k-2}$.

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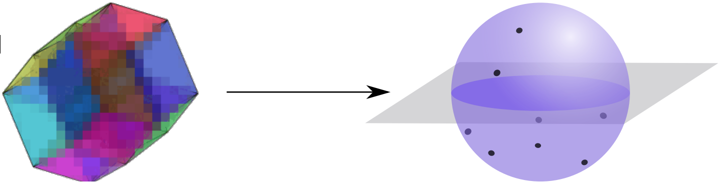
4-dimensional polytope with 8 facets



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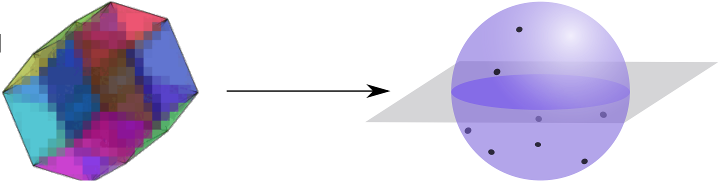


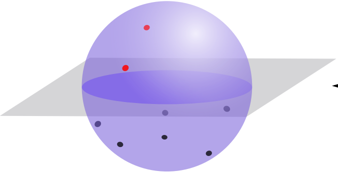
The *combinatorial type* of a polytope is the information of which sets of facets nontrivially intersect, and it is encoded in the Gale diagram by

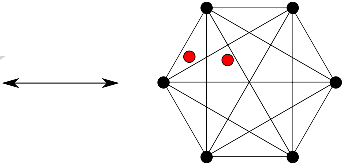
$$\bigcap_{j \in J} f_j \neq \emptyset \iff 0 \in \text{conv}(\{p_i : i \notin J\})$$

Geometric Lens: Affine Gale Diagrams

We can reduce the dimension of the Gale diagram by one via taking a signed projection to a hyperplane. The resulting signed set of points is an *affine Gale diagram* for the polytope.

- Note: this construction depends on the choice of hyperplane.





Lemma (B.)

Every compact Coxeter d-polytope with $d + 4$ facets has an affine Gale diagram with points in general position and with exactly 2 positive points that lie in the convex hull of the negative points.

How can we obtain a list of possible combinatorial types?

Geometric Lens: Point Set Order Types

Definition

The *order type* of a set of points is the information of the orientation (clockwise, counterclockwise, or collinear) of every ordered triple of points.

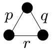
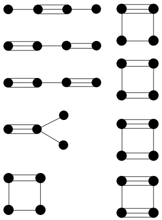
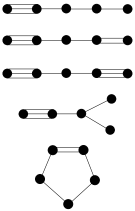
- The order types of point sets containing $<= 11$ points were classified by Aichholzer, Aurenhammer, and Krasser (2002).

By iterating over order types and choices of positive points for affine Gale diagrams, we obtain a list of combinatorial types of simple d-polytopes with $d + 4$ facets and at least two pairs of non-intersecting facets.

Dimension	**Combinatorial Types**	**Types Realized by Coxeter Polytopes**
4	34 (Grünbaum, Sreedharan)	14
5	186	6
6	265	?

Weighted Graph Lens: Lannér Diagrams

- From the combinatorial type, we can determine a list of minimal non-faces, also known as *missing faces*.
- There is a limited list of Coxeter subdiagrams corresponding to missing faces. These were classified by Lannér in 1950 and are called *Lannér diagrams*.

order	2	3	4	5
diagrams		 where $2 \leq p, q, r \in \mathbb{N}$ and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$		

Since there are relatively few Lannér diagrams on 2, 4, or 5 vertices, we can start to "build" up Coxeter diagrams by placing Lannér diagrams.

Algebraic Lens: Gram Matrices

Definition

The *Gram matrix* M of a polytope with facets $f_1, \ldots, f_n$ is given by

$$M_{ii} = 1 \text{ and } M_{ij} = \begin{cases} -\cos(\theta) & \text{if } \angle f_i f_j = \theta \\ -\cosh(\text{dist}(f_i, f_j)) & \text{if } f_i \text{ and } f_j \text{ diverge} \end{cases}$$

- A $n \times n$ matrix is the Gram matrix of a d-polytope if and only if its entries are in the proper ranges and it has signature $(d, 1, n - d - 1)$, i.e., rank $d + 1$ and exactly one negative eigenvalue.
- Most entries are determined by the dihedral angles.
- Remaining entries can be viewed as variables in a system of equations given by the rank condition.

Classifying d-Polytopes with $d + 4$ Facets

Procedure for classification:

1. Fix a combinatorial type for the polytope.
2. Select a set of dihedral angles to take values at most $\pi/6$ (restricted by combinatorial type). Iterate over all such choices.
3. Assign dihedral angles within missing faces of size 4 and 5 (limited by the set of Lannér diagrams).
4. Assign the remaining dihedral angles of size at least $\pi/5$.
5. Solve (using, e.g., Mathematica) for the remaining Gram matrix entries using the rank condition.

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Theorem (B. 2021, Ma–Zheng 2022)

There are 348 hyperbolic Coxeter 4-polytopes with 8 facets, and 51 hyperbolic Coxeter 5-polytopes with 9 facets.

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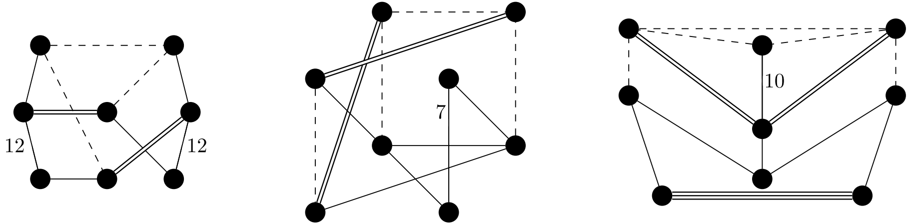
Dimension 6? We don't expect many new polytopes.

Interesting Examples



The classification includes the only known

- polytope in dimension higher than 3 with a dihedral angle of less than $\frac{\pi}{10}$,
- polytope in dimension higher than 3 with a dihedral angle of $\frac{\pi}{7}$, and
- polytopes in dimension 5 with a dihedral angle of less than $\frac{\pi}{6}$.

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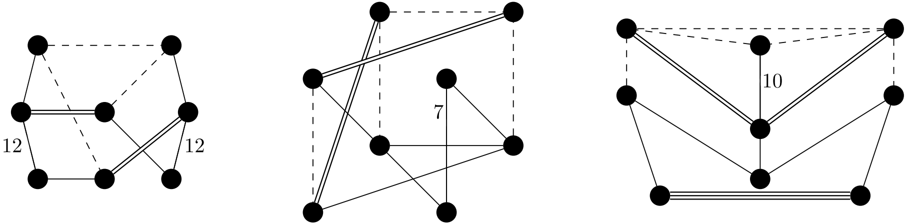

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How is this classification useful?

History of Dimension Bounds

- Vinberg proved in 1984 that compact Coxeter polytopes do not arise in dimensions higher than 29.
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  - We know of two examples in dimension 7, having 11 and 13 facets.
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  - We know of two examples in dimension 7, having 11 and 13 facets.
  - These examples are all due to work of Bugaenko in 1992.
- There are compact Coxeter polytopes with arbitrarily many facets in dimension 6.
  - Proved by Allcock in 2009 by repeatedly gluing copies of one of Bugaenko's polytopes together.

Improved Dimension Bounds

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Theorem (B.)

A compact Coxeter d-polytope with $d + k$ facets has dimension bounded above by

k	**5**	**6**	**7**	**8**	**9**	**10**
Dimension Upper Bound	9	12	15	18	22	26

Thank you!

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